The Zariski topology on Spec(R)

This is a series of exercises. As usual, if R is a (commutative) ring, we let Spec(R) denote the set of prime ideals of R - we will define a topological space on this set. This could have been done (should have been done) in the Commutative Algebra Class, but time was a little tight. While it is not crucial for the Homological Algebra class, it will provide an interesting example. I will not collect it, but try to get it done by the start of the second week of class.

(1) For each subset of $E \subseteq R$, let $V(E)$ denote the set of all prime ideals of R that contain E. Prove that
  (a) if I is the ideal generated by E, then $V(E) = V(I) = V(\sqrt{I})$,
  (b) $V(0) = \mathrm{Spec}(R)$, $V(1) = \emptyset$,
  (c) if $\{I_j\}_{k\in K}$ is a family of ideals of R, then
$$V(\bigcup_{k\in K} I_k) = \bigcap_{k\in K} V(I_k),$$
  (d) $V(I \bigcap J) = V(IJ) = V(I) \bigcup V(J)$ for any ideals I, J of R.

  The above results show that the sets $\{V(E)\}$ satisfy the axioms for closed sets in a topological space. The resulting topology is called the *Zariski topology* on Spec(R).

(2) For each $a \in R$, let $U(a) = \mathrm{Spec}(R) \setminus V(a)$. Thus the $U(a)$ are open sets in the Zariski topology. Show that this collection $\{U(a)\}$ forms a basis of open sets for the Zariski topology, and that
  (a) $U(a) \cap U(b) = U(ab)$;
  (b) $U(a) = \emptyset \Leftrightarrow a$ is nilpotent;
  (c) $U(a) = \mathrm{Spec}(R) \Leftrightarrow a$ is a unit;
  (d) Spec(R) is compact.

(3) Let $\varphi : R \longrightarrow T$ be a ring homomorphism. Then there is an induced map φ^* $\mathrm{Spec}(T) \longrightarrow \mathrm{Spec}(R)$, given by $\varphi^*(P) = \varphi^{-1}(P)$. Show that φ^* is continuous. (Thus the assignment $R \rightsquigarrow \mathrm{Spec}(R)$ is a contravariant functor from the category of commutative rings to the category of topological space.)

(4) Let R be a ring. Then Max(R) denotes the set of maximal ideals of R. It becomes a topological space when viewed as a subset of Spec(R). Let X be a compact Hausdorff space. Let $R = C(X)$. Prove that Max(R) is homeomorphic to X. (This is actually a non-trivial result that requires Urysohn's Lemma. Given time I may do it in class.)