Homological Algebra **Solutions to HW #4** **Spring 2015**

(1) Let $R \to T$ be a ring homomorphism. Let M be a flat left R-module. Show that $T \otimes_R M$ is a flat left T-module.

Proof. First note that if A is any right T-module, then $A \otimes_T (T \otimes_R M) \cong (A \otimes_T T) \otimes_R M \cong A \otimes_R M$. Now consider the following short exact sequence of right T-modules:

$$0 \to A \to B \to C \to 0$$

Apply $\square \otimes_T (T \otimes_R M)$ to the sequence. Using the isomorphism from the first sentence, we get

$$0 \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0$$

which is exact since by assumption M is a flat left R-module. Hence $T \otimes_R M$ is a flat (left) T-module. $\square$

(2) Let p be a prime integer and set $\mathbb{Z}(p^\infty) := \{a/p^n + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z} a \in \mathbb{Z}\}$. Then this is a $\mathbb{Z}$ submodule of $\mathbb{Q}/\mathbb{Z}$ (you do not need to show this, though it is easy). Show that $\mathbb{Z}(p^\infty)$ is an injective R-module. Hint: It suffices to show that $\mathbb{Z}(p^\infty)$ is a divisible module, since $\mathbb{Z}$ is a PID. To that end, let $t \in \mathbb{Z}$ and $x := a/p^n + \mathbb{Z} \in \mathbb{Z}(p^\infty)$. To show that x is divisible by t, write $t = bp^s$ where b is relatively prime to p. It suffices to show that b and p^s separately "divide" each element of the group. Clearly the latter element does. Then use the fact that b is relatively prime to p^n to show it "divides" $a/p^n + \mathbb{Z}$.

Proof. My hint mostly gives this one away. Let $a/p^n + \mathbb{Z}$ be an arbitrary element of $\mathbb{Z}(p^\infty)$ and let $t \in \mathbb{Z}$. Write $t = bp^s$ where b is relatively prime to p. It will suffice to show that there exists $c/p^m + \mathbb{Z} \in \mathbb{Z}(p^\infty)$ such that $b(c/p^m + \mathbb{Z}) = a/p^{n+s} + \mathbb{Z}\ *$.

Since b and p^{n+s} are relatively prime, there exists $x, y \in \mathbb{Z}$ such that $xb + yp^{n+s} = 1$. Dividing both sides by p^{n+s}, we get

$$\frac{xb}{p^{n+s}} + y = \frac{1}{p^{n+s}}.$$

Now multiply both sides by a and rearrange and we get

$$\frac{axb}{p^{n+s}} - \frac{a}{p^{n+s}} = -ya$$

Thus the RHS is an integer. Therefore the equation in $*$ will be satisfied if we let $c = xb$ and $m = n + s$. $\square$

(3) Let $M_1 \subset M_2 \subset \ldots \subset M_n \subset \ldots$ be an ascending chain of submodules of M. Prove that

$$\lim_{\rightarrow} M_i = \bigcup M_i.$$

Proof. We will show that $\bigcup M_i$ satisfies the universal mapping property. We may as well assume that $M = \bigcup M_i$. Clearly for each $i \in \mathbb{N}$, there is a map $M_i \to M$, namely inclusion, that makes all the necessary diagrams commute. Suppose that there is a module X and maps $f_i : M_i \to X$ for each $i \in \mathbb{N}$, where all the triangles

$$\begin{array}{ccc} M_j & \xrightarrow{f_j} & X \\ \uparrow & \nearrow & \\ M_i & & \end{array}$$

commute, when $i < j$.

We need to define a map $f : M \to X$ that makes all the appropriate maps commute. Let $m \in M$. Thus $m \in M_i$ for some i. Set $f(m) = f_i(m)$. Since all the above triangles commute, this map is well defined (i.e., it does not matter which $i \in \mathbb{N}$ is chosen, as long as $m \in M_i$). Clearly this f works.

□

(4) Let k be a field and let J be the ideal (x) in $k[x]$. Consider the inverse system in the category of commutative rings given by $\{R_n := k[x]/J^n, \varphi_{ji}\}$ where for $j > i$, $\varphi_{ji} : k[x]/J^j \to k[x]/J^i$ is the canonical projection (note: $J^j \subset J^i$). Prove that

$$\lim_{\leftarrow} R_n = k[[x]], \text{ the power series ring.}$$

Note that in fact, k can be any commutative ring

Proof. First we define a map $f_n : k[[x]] \to R_n$, by taking the natural projection of $k[[x]]$ onto $k[[x]]/x^n k[[x]] \cong R_n$ for each n. Now suppose that there is a ring T and ring maps $g_n : T \to R_n$ for each n. We have to construct a ring homomorphism $g : T \to k[[x]]$ such that for each n, $f_n \circ g = g_n$.

Let $t \in T$. Then $g(t) = \sum_{i=0}^{\infty} a_i x^i$ must be a power series in x over k. For each i we define a_i to be the lead coefficient of $g_{i+1}(t) \in R_{i+1}$. Since $\varphi_{mn} \circ g_n = g_m$ whenever $m > n$, g is well defined. It is easy to check that $g(t + t') = g(t) + g(t')$. To see that $g(tt') = g(t)g(t')$ is a little more work to write down, but it follows from the fact that each coefficient of $g(t)g(t')$ is defined by a finite set of coefficients of each of $g(t)$ and $g(t')$.

□