(1) Let $R = \mathbb{Z}/p^n\mathbb{Z}$, where $p$ is a prime number. Show that $R$ is injective as a module over itself.

**Proof:** We use Baer's criterion. First note that all ideals of $R$ are subgroups of $R$. Hence the ideals are linearly ordered and principal. Moreover each ideal is generated by $\bar{p}^k$ for some $k < n$ (here $\bar{p}$ denotes the image of $p$ in $R$). Let $f : I \to R$ be an $R$ homomorphism. We have to show that $f$ can be extended to a map $\hat{f} : R \to R$. Notice that $\text{ann}(\bar{p}^k) = (\bar{p}^{n-k})$. Thus, if $a = f(\bar{p}^k)$, it follows that $\text{ann}(a) \supseteq (\bar{p}^{n-k})$. In particular $a$ is an element of the form $u\bar{p}^t$, where $u$ is a unit of $R$ and $t \geq k$. Thus define $\hat{f}$ on all of $R$ by sending $1 \mapsto up^{t-k}$. Hence we see that every map from an ideal of $R$ to $R$ extends to all of $R$. Thus by Baer we are done.

(2) (3.31) (i) Let $P$ be the set of all prime numbers. Prove that $\bigoplus_{p\in P} \mathbb{Z}_p$ is the torsion subgroup of $\prod_{p\in P} \mathbb{Z}_p$.

**Proof:** The key to this problem and the next is to note that for any $n \in \mathbb{Z}$, either $n$ is a unit or zero in $\mathbb{Z}_p$. First let $t \in \bigoplus_{p\in P} \mathbb{Z}_p$. Let $\{p_1, p_2, \ldots, p_n\}$ be the coordinates where $t$ is not zero. Let $n$ be the product of these prime numbers. Then $nt = 0$. Thus $\bigoplus_{p\in P} \mathbb{Z}_p \subseteq \tau(\prod_{p\in P} \mathbb{Z}_p)$. For the reverse direction. Let $x \in \tau(\prod_{p\in P} \mathbb{Z}_p)$. Then $nx = 0$ some integer $n$. Let $S = \{p_1, p_2, \ldots, p_n\}$ be the set of prime divisors of $n$. If $x$ is not in the direct sum, then there is a prime $q \notin S$, such that the $q$th coordinate of $x$, $x_q$, is not zero. But then $nx_q \neq 0$, a contradiction. Thus we have containment in the reverse direction - done.

(3) (3.31) (ii) Prove that $M = \prod_{p\in P} \mathbb{Z}_p / \bigoplus_{p\in P} \mathbb{Z}_p$ is a divisible abelian group.

**Proof:** Let $0 \neq x = (x_q) \in M$. Let $n \in \mathbb{Z}$. Let $S = \{p_1, p_2, \ldots, p_n\}$ be the set of prime divisors of $n$. Notice that for $q \in P \setminus S$, $n$ is a unit in $\mathbb{Z}_q$. Let $y = (y_q) \in M$ be defined coordinate wise as $y_q = n^{-1}x_q \bmod q$ for $q \in P \setminus S$, while $y_q = 0$ otherwise (note we do not care what happens to $y$ at a finite number of coordinates). Thus $ny = x$ - done.

(4) (3.39) (i) Prove that $\mathbb{Q}$ is a flat but not faithfully flat $\mathbb{Z}$ modules.

**Proof**: Since $\mathbb{Q}$ is torsion free and $\mathbb{Z}$ is a PID, the fact that $\mathbb{Q}$ is torsion free assures us is is flat. We also know that since it is divisible, $\mathbb{Q} \otimes \mathbb{Z}_n = 0$ for any $n$. Hence it is not faithfully flat.

(5) (3.39) (ii) Prove that an abelian group $G$ is a faithfully flat $\mathbb{Z}$ module if and only if it is torsion free and $pG \neq G$ for all primes $p$.

**Proof**: Assume $G$ is faithfully flat. Then since $R$ is a domain, $G$ is torsion free. The map $\mathbb{Z} \to \mathbb{Z}$ via multiplication by $p$ is a monomorphism. As $G$ is faithfully flat (and so torsion free) we have the diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \to & G \otimes \mathbb{Z} & \to & G \otimes \mathbb{Z} & \to & G \otimes \mathbb{Z}_p & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & G & \stackrel{p}{\to} & G & \to & G/pG & \to & 0 \end{array}$$

where the first two horizontal maps are isomorphisms, and so the last horizontal map is an isomorphism (five lemma). Since $G$ is faithfully flat $G \otimes \mathbb{Z}_p \neq 0$. Hence $G/pG \neq 0$, equivalently $pG \neq G$.

Conversely, let $G$ be a torsion free abelian group such that $pG \neq G$ for all primes $p$. Since $\mathbb{Z}$ is a PID, any torsion free module is flat. In particular $G$ is flat. Hence we only need show that $G \otimes M \neq 0$ for all nonzero abelian groups $M$. Since $M \neq 0$, $M$ contains a nonzero cyclic abelian group $H$. Moreover any cyclic abelian group maps onto the group $\mathbb{Z}_p$ for some $p$ a prime number. Since $G$ is flat and $pG \neq G$, it follows from the above diagram that $G \otimes \mathbb{Z}_p \neq 0$. But then $G \otimes H$ maps onto the nonzero module $G \otimes \mathbb{Z}_p$ (by general tensor facts). Thus $G \otimes H \neq 0$. As $G$ is flat, $G \otimes H \subseteq G \otimes M$, which implies that the latter module is nonzero and we are done.

(6) Let $R$ be an integral domain. Show that an ideal $I$ of $R$ is invertible if and only if it is projective as a module over $R$.

**Proof**: First suppose that $I$ is projective. We want to show that $1 \in II^{-1}$. Let $\{\phi_k\}_{k \in K}$ and $\{a_k\}_{k \in K} \subset I$ be a projective basis of $I$. Then, since $Q := \mathrm{Frac}(R)$ is an injective $R$-module, each $\phi_k$ can be extended to map $R \to Q$. But each such map is multiplication by some $q_k \in Q$ (just see where 1 maps to). Thus $\phi_k(x) = q_k x$ for all $x \in I$. Hence $q_k \in I^{-1}$ for all $k$. Moreover, by the definition of a projective basis, it follows that $q_k \neq 0$ for only finitely many $k$ (why is that?) Thus $\{\phi_k\}_{k \in K}$ is finite. Then by definition of a projective basis for any $x \in I$ we have

$$x = \phi_1(x)a_1 + \ldots + \phi_k(x)a_k = xq_1a_1 + \ldots + xq_ka_k$$

We can then factor out the $x$ on the right side and cancel (we are in a domain after all) to get

$$1 = q_1a_1 + q_2a_2 + \ldots + q_ka_k$$

Since $q_j \in I^{-1}$ and $a_j \in I$ we are done with this direction.

For the converse, suppose that $I$ is invertible. Say $1 = q_1a_1 + \ldots + q_ka_k$ for $q_j \in I^{-1}$ and $a_j \in I$. It follows that multiplication by each $q_j$ defines a $\phi_j : I \to R$ (the image is in $R$, since $q_j \in I^{-1}$). It is now easy to see (check!) that the sets $\{\phi_j\}$ and $\{a_j\}$ form a projective basis for $I$, which completes the proof.