(1) (2.28) Let R be a domain with $Q = \mathrm{Frac}(R)$ its quotient field. If A is an R-module, show that every element of $Q \otimes_R A$ has the form $q \otimes a$ for some $q \in Q$ and $a \in A$.

Proof. An arbitrary element of $Q \otimes_R A$ has the form

$$q_1 \otimes a_1 + q_2 \otimes a_2 + \cdots + q_n \otimes a_n, q_i \in Q \text{ and } a_i \in A.$$

Then each $q_i = b_i/c_i$ for $b_i, c_i \in R$. Let c be the product of the c_i. Hence each $q_i = d_i/c$ for some $d_i \in R$. Moreover $q_i \otimes a_i = (d_i/c) \otimes a_i = (1/c)d_i \otimes a_i = (1/c) \otimes d_ia_i$. Hence $\sum q_i \otimes a_i = \sum(1/c) \otimes d_ia_i$. By the definition of tensor, $\sum(1/c) \otimes d_ia_i = (1/c) \otimes (\sum d_ia_i)$ - done. □

(2) (2.29(iii)) Let m and n be positive integers and let $d = (n, m)$. Prove that there is an isomorphism of abelian groups $\mathbb{Z}_n \otimes \mathbb{Z}_m \cong \mathbb{Z}_d$.

Proof. Consider the short exact sequence $0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}_n \to 0$, where the map n is multiplication by n. Apply $- \otimes \mathbb{Z}_m$ to get

$$\mathbb{Z} \otimes \mathbb{Z}_m \xrightarrow{n} \mathbb{Z} \otimes \mathbb{Z}_m \to \mathbb{Z}_n \otimes \mathbb{Z}_m \to 0$$

We know that $\mathbb{Z} \otimes \mathbb{Z}_m \cong \mathbb{Z}_m$. Moreover the first map is still multiplication by n (trace through the isomorphisms). Thus $\mathbb{Z}_n \otimes \mathbb{Z}_m \cong \mathbb{Z}_m/n\mathbb{Z}_m$, which by (ii) is isomorphic to $\mathbb{Z}_d$. □

(3) (2.31) Assume that the following diagram commutes and that the vertical arrow are isomorphisms.

$$\begin{array}{ccccccccc} 0 & \to & A' & \xrightarrow{f_1} & A & \xrightarrow{g_1} & A'' & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & B' & \xrightarrow{f_2} & B & \xrightarrow{g_2} & B'' & \to & 0 \end{array}$$

Prove that the top row is exact if and only if the bottom row is exact.

Proof. Assume that the top row is exact. We must show that f_2 is injective, g_2 is surjective, and $\ker(g_2) = \mathrm{im}(f_2)$. Let $b' \in \ker(f_2)$. Let a' be its image in A'. Thus $f_1(a')$ maps to zero in A. Since this vertical map is an isomorphism, we must have $f_1(a') = 0$. But f_1 is injective by assumption, thus $a' = 0$, Since the vertical map is an iso, $b' = 0$ and so f_2 is injective.

Let $b'' \in B''$. Let a'' be its image in A''. Since g_1 is surjective, we can pullback to an element of A. Let $a \in A$ be such that $g_1(a) = a''$. Then a has a unique image in b in B. Since the diagram commutes, $g_2(b) = b''$. Since b'' was arbitrary, g_2 is surjective.

It is also easy to see that $\ker(g_2) = \mathrm{im}(f_2)$ (translation: this is tedious and I am tired of writing it up).

For the other direction, just put the bottom row on top. Since the vertical maps were isomorphisms, they are reversible. □

(4) (3.11) Prove that $\mathrm{Hom}_R(P, R) \neq \{0\}$ if P is a projective left R-module.

Proof. This will follow easily from the existence of a projective basis for P. Specifically, let $\{a_i\}$, $a_i \in P$ and $\{\varphi_i\}$, $\varphi_i \in \mathrm{Hom}(P, R)$ be a projective basis of P. Now suppose that $\mathrm{Hom}(P, R) = 0$, then each $\varphi_i = 0$. Hence for each $x \in P$, we have $x = \sum \varphi_i(x)a_i = 0$, i.e., $P = 0$. Done. □

(5) (3.12) If P is finitely generated, prove that P is projective if and only if $1_P \in \mathrm{im}\ \nu$, where $\nu : Hom_R(P, R) \otimes_R P \to Hom_R(P, P)$ is defined, for all $x \in P$, by $f \otimes x \mapsto \tilde{f}$, where $\tilde{f} : y \mapsto f(y)x$.

Proof. First suppose that P is finitely generated. We show that P has a finite projective basis. In fact it follows from the proof that P has a projective basis, but we show the proof. There exists a finitely generated free module $F = R^n$ such that there exists a short exact sequence:

$$0 \to K \to F \to P \to 0$$

Since P is projective, this sequence splits. Hence $F \cong P \oplus K$. Now let $e_i \in F$ have a 1 in the i-coordinate and 0 everyplace else and let $\tilde{\varphi}_i, i = 1, \ldots, n$ be the natural projections of F onto its i-th coordinate. Let $\varphi_i \in \mathrm{Hom}(P, R)$ be the restriction of $\tilde{\varphi}$ to P and let $e_i = x_i + k_i$ where $x_i \in P$ and $k_i \in K$. Hence $\{x_i\}$ and $\{\varphi_i\}$ forms a finite projective basis of P and $\sum(\varphi_i \otimes x_i) \in \mathrm{Hom}_R(P, R) \otimes_R P$. Then we claim that $\nu(\sum(\varphi_i \otimes x_i))$ is the identity map on P, which proves this direction of the result. To see why observe that for $y \in P$ $\nu(\sum(\varphi_i \otimes x_i))(y) = \sum \varphi_i(y)x_i$. And by definition of a projective basis, this last term equals y. Thus $\nu(\sum(\varphi_i \otimes x_i)) = 1_P$, and this direction of the proof is done.

Now suppose that $1_P \in \mathrm{im}\ \nu$. Say $1_P = \nu(\sum(f_i \otimes z_i)$. Hence for any $y \in P$, $y = \nu(\sum(f_i \otimes z_i)(y)$. But $\nu(\sum(f_i \otimes z_i)(y) = \sum f_i(y)x_i$, which shows that $\{x_i\}$ and $\{f_i\}$ forms a projective basis of P. □