(1) (2.2) Give an example of a left R-module $M = S \oplus T$ having a submodule N such that $N \neq (N \cap S) \oplus (N \cap T)$.

Proof: First you have to understand what is meant by $(N \cap S) \oplus (N \cap T)$. You have to view S as the submodule of M via $S \oplus 0$, i.e., S is the set of al elements of M whose second coordinate is 0 (similarly for T as a submodule of M). Let $R = \mathbb{Z}$, let $S = T = \mathbb{Z}$. Let N be the diagonal of M, i.e., $N = \{(a, a) : a \in \mathbb{Z}\}$. Since the only element of N whose second coordinate is 0, it follows that $N \cap S = 0$ similarly $N \cap T = 0$. Thus $N \neq (N \cap S) \oplus (N \cap T) = 0$.

(2) (2.7) Use the left exactness of Hom to prove that if G is an abelian group, then $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, G) \cong G[n]$, where $G[n] := \{g \in G : ng = 0\}$.

Proof: We have the exact sequence:

$$0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \mathbb{Z}_n \to 0,$$

where f is multiplication by n and g is the natural projection. We then get the exact sequence:

$$0 \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_n, G) \xrightarrow{g^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, G) \xrightarrow{f^*} \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, G).$$

Recall that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, G) \cong G$ (also see the next problem) via the map that sends $h \mapsto h(1)$. Moreover it is easy to see that the map f^* multiplies G by n. Thus by definition $\ker f^* = G[n]$. Since g^* is monic and im $g^* = \ker f^*$, we have the result.

(3) (2.13) Let M be a left R-module. Prove that the map $\varphi_M : \mathrm{Hom}_R(R, M) \to M$ given by $f \mapsto f(1)$ is an R-isomorphism.

Proof: First note that $\mathrm{Hom}_R(R, M)$ is a left R-module since R is an $R - R$ bimodule. The rest is straight forward checking of the axioms.

(4) (2.17) If $0 \to A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D \xrightarrow{k} D$ is exact, prove that there is an exact sequence

$$0 \to \text{coker} f \xrightarrow{\alpha} C \xrightarrow{\beta} \text{ker} k \to 0$$

where $\alpha : b + \text{im} f \mapsto g(b)$ and $\beta : c \mapsto h(c)$.

Proof: This is trivial, it just tests your understanding of what is going on. First we show that α is injective. But if $\alpha(b + \text{im} f) = 0$, then $g(b) = 0$. Hence $b \in \text{ker} g = \text{im } f$. Thus $b + \text{im} f = 0$.

Since the im $\alpha =$ im g and im $g =$ ker h, we see that im $\alpha \subseteq$ ker β. Conversely, it is clear that ker $\beta =$ ker h. Thus we have equality and we are done.