Math 629-Homework (Spring 2015) # 4 – Due ???

Hand-in

(1) Let $R \to T$ be a ring homomorphism. Let M be a flat left R-module. Show that $T \otimes_R M$ is a flat left R-module.

(2) Let p be a prime integer and set $\mathbb{Z}(p^\infty) := \{a/p^n + \mathbb{Z} \in \mathbb{Q}/\mathbb{Z} a \in \mathbb{Z}\}$. Then this is a $\mathbb{Z}$ submodule of $\mathbb{Q}/\mathbb{Z}$ (you do not need to show this, though it is easy). Show that $\mathbb{Z}(p^\infty)$ is an injective R-module. Hint: It suffices to show that $\mathbb{Z}(p^\infty)$ is a divisible module, since $\mathbb{Z}$ is a PID. To that end, let $t \in \mathbb{Z}$ and $x := a/p^n + \mathbb{Z} \in \mathbb{Z}(p^\infty)$. To show that x is divisible by t, write $t = bp^s$ where b is relatively prime to p. It suffices to show that b and p^s separately "divide" each element of the group. Clearly the latter element does. Then use the fact that b is relatively prime to p^n to show it "divides" $a/p^n + \mathbb{Z}$.

(3) Let $M_1 \subset M_2 \subset \ldots \subset M_n \subset \ldots$ be an ascending chain of submodules of M. Prove that

$$\lim_{\rightarrow} M_i = \bigcup M_i.$$

(4) Let k be a field and let J be the ideal (x) in $k[x]$. Consider the inverse system in the category of commutative rings given by $\{R_n := k[x]/J^n, \varphi_{ji}\}$ where for $j > i$, $\varphi_{ji} : k[x]/J^j \to k[x]/J^i$ is the canonical projection (note: $J^j \subset J^i$). Prove that

$$\lim_{\leftarrow} R_n = k[[x]], \text{ the power series ring.}$$

Note that in fact, k can be any commutative ring