Math 621 **HW #10** **– Due Monday, May 2.**

1. An abelian group G (under addition) is called an *ordered group* if the there is a total ordering on the elements of G such that if $x \leq y$, then $a + x \leq b + x$ for all $x, y, z \in G$. Let K be a field. A valuation on K is an onto map $v : K^{\times} \to G$, where G is an ordered group, such that

   i) $v(ab) = v(a) + v(b)$ (i.e., v is a group homomorphism.)

   ii) $v(x + y) \geq \min\{v(x), v(y)\}$ for all $x, y \in K^{\times}$ with $x + y \neq 0$.

   Let $R_v = \{x \in K : v(x) \geq 0\}$. Then R is a subring of K. You do not have to prove this (the proof is exactly the same as for a discrete valuation.) Moreover the units of R_v are the elements $\{x \in R_v : v(x) = 0\}$ (also same proof).

   (a) Show that the set $M = \{x \in R_v : v(x) > 0\}$ is a maximal ideal of R_v and the only maximal ideal of this ring.

2. Let $G = \mathbb{Z} \times \mathbb{Z}$ with the lexicographic or dictionary ordering. Thus $(a, b) > (c, d)$ if $a > c$ or if $a = c$ and $b > d$. This is how words in the dictionary are totaly ordered. So for example $(1, -5) > (0, 10)$. Let $S = \mathbb{Q}[x, y]$. Let K be the quotient field of S, i.e., all rational polynomials in two variables, with coefficients in $\mathbb{Q}$. We will define a valuation v on K to G, by first defining it on S. First for a monomial $qx^iy^j \in S$ ($q \in \mathbb{Q}$), we set $v(qx^iy^j) = (i, j)$. For arbitrary $f(x, y) \in S$, f is of course a sum of monomials g_k. We define $v(f) = \min\{v(g_j)\}$. Thus if $f = x^3y^4 + 5x^2y^7$, then $v(f) = \min\{v(x^3y^4), v(5x^2y^7)\} = \min\{(3, 4), (2, 7)\} = (2, 7)$. Now extend v to all of K by defining $v(f/h) = v(f) - v(h)$. One checks that this defines a valuation on K. You should convince yourself that this is true, but you do not have to hand in a proof of this (it is too tedious). The ring R_v is a subring of K that clearly contains S, but it is bigger than S.

   (b) Determine if the following elements are in R_v:
   $(x^2 + y^3)/(x + y^4)$, $\qquad$ $(x^3 + x^2y^2)/(xy + y^3)$.

   (c) Let $P = \{h \in R_v : v(h) > (0, n)$ for all positive integers $n\}$. Show that P is an ideal of R_v and that it cannot not be generated by finitely many elements. (Hint: For the second part note that $(1, -n) > (0, n)$, for all $n \in \mathbb{Z}^+$.)