Grade: ________ KEY

[18pts] 1. The following questions are short answer and only require a brief explanation.

(a) Find the dimension of the subspace of $\mathbb{R}^2$ spanned by the vectors $\begin{bmatrix} 1 \\ -5 \end{bmatrix}, \begin{bmatrix} 2 \\ -10 \end{bmatrix}, \begin{bmatrix} -3 \\ 15 \end{bmatrix}$.

$\begin{pmatrix} 2 \\ 10 \end{pmatrix} = 2\begin{pmatrix} 1 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} -3 \\ 15 \end{pmatrix} = -3\begin{pmatrix} 1 \\ -5 \end{pmatrix}$

So $\boxed{\text{dimension equals } 1}$

(b) $A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 3 & 3 \\ 2 & 6 & 5 \end{pmatrix}$ row reduces to $\begin{pmatrix} 1 & 3 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Find a basis for the row space of A.

$\{(1\,3\,2), (0\,0\,1)\}$

(c) Let A be a 3×5 matrix. If the nullspace of A has dimension 4, what is the rank of A? (What Theorem are you using?)

$\dim \operatorname{Nul}(A) + \operatorname{Rank}(A) = \#$ of columns

So $\operatorname{Rank}(A) = 5 - 4 = \boxed{1}$

[10pts] 2. Let H be the set of all vectors of the form $\begin{bmatrix} 2b + 3c \\ -b \\ 2c \end{bmatrix}$, where b and c are arbitrary real numbers. Find vectors $\mathbf{u}$ and $\mathbf{v}$ such that $H = \text{Span}\{\mathbf{u},\mathbf{v}\}$.

$u = \begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}, \quad v = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix}$

[8pts] 3. Determine if the set H of all matrices of the form $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$ is a subspace of $M_{2\times 2}$ (the space of all 2×2 matrices).

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} + \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = \begin{pmatrix} a+x & b+y \\ 0 & d+z \end{pmatrix} \text{ is upper triangular}$$

$$\alpha \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ 0 & \alpha d \end{pmatrix} \text{ is upper triangular}$$

So H is a subspace

[8pts] 4. Let A be a 3×3 matrix, let x be an unknown 3-tuple and let b be a vector in $\mathbb{R}^3$. If $Ax = b$ has a unique solution, explain why $Ax = v$ has a solution for all v in $\mathbb{R}^3$.

Since $Ax = b$ has a unique solution the augmented matrix $(A|b)$ has unique solution. Thus A has a pivot in each column. Since it is 3x3, it has a pivot in each row. So $(A|b)$ always has a solution

[10pts] 5. Find a basis for the space spanned by the vectors $\begin{bmatrix} 2 \\ 0 \\ -4 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$.

$$\begin{pmatrix} 2 & -4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ -4 & 8 & 2 & 2 \end{pmatrix} \xrightarrow{2} \begin{pmatrix} 2 & -4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 4 & 8 \end{pmatrix} \xrightarrow{-4}$$

$$\begin{pmatrix} 2 & -4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 4 \end{pmatrix}$$

Thus columns 1, 3 and 4 have pivots. So columns

$$\begin{pmatrix} 2 \\ 0 \\ -4 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \text{ span}$$

[10pts] 6. Show that the polynomials $1+2t^3, 2t, -2+4t^2, -12t+8t^3$ form a basis of $\mathbb{P}_3$.

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 2 & 0 & 0 & 8 \end{pmatrix} \xrightarrow{-2} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 4 & 8 \end{pmatrix} \xrightarrow{-1} \begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 2 & 0 & -12 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}$$

Thus every row and every column has a pivot. So

$\boxed{\text{original rows are lin. ind. and span}}$

7. Let $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$ and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, where $\mathbf{b}_1 = \begin{bmatrix} -1 \\ 8 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 1 \\ -7 \end{bmatrix}$, $\mathbf{c}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, and $\mathbf{c}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then $\mathcal{B}$ and $\mathcal{C}$ are bases of $\mathbb{R}^2$.

[10pt] (a) Find the change of coordinate matrix from $\mathcal{B}$ to $\mathcal{C}$.

$$\left(\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 2 & 1 & 8 & -7 \end{array}\right) \xrightarrow{-2} \left(\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 0 & -1 & 10 & -9 \end{array}\right)_{\cdot -1} \rightarrow \left(\begin{array}{cc|cc} 1 & 1 & -1 & 1 \\ 0 & 1 & -10 & 9 \end{array}\right)_{-1}$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 9 & -8 \\ 0 & 1 & -10 & 9 \end{array}\right)$$

$\boxed{\text{So } \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{pmatrix} 9 & -8 \\ -10 & 9 \end{pmatrix}}$

[8pt] (b) If $v = 4\mathbf{b}_1 - 2\mathbf{b}_2$, find $[v]_\mathcal{C}$.

$[v]_\mathcal{B} = \begin{pmatrix} 4 \\ -2 \end{pmatrix}$

So $[v]_\mathcal{C} = \underset{\mathcal{C}\leftarrow\mathcal{B}}{P} \begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 9 & -8 \\ -10 & 9 \end{pmatrix}\begin{pmatrix} 4 \\ -2 \end{pmatrix} = \begin{pmatrix} 36+16 \\ -40-18 \end{pmatrix}$

$\boxed{= \begin{pmatrix} 52 \\ -58 \end{pmatrix}}$

[18pt] 8. Suppose that in a small town 40% of the people walk to work, while the rest drive to work. Suppose that each year the proportion of people that either walk or drive to work changes according to the stochastic matrix

$$\begin{array}{c|cc} & \text{Walk} & \text{Drive} \\ \hline \text{Walk} & .7 & .4 \\ \text{Drive} & .3 & .6 \end{array}$$

(a) Next year what percentage of the town will walk to work?

$$\begin{pmatrix} .7 & .4 \\ .3 & .6 \end{pmatrix}\begin{pmatrix} .4 \\ .6 \end{pmatrix} = \begin{pmatrix} .28+.24 \\ .12+.36 \end{pmatrix} = \begin{pmatrix} .52 \\ .48 \end{pmatrix}$$

52% will walk to work next year

(b) In the long run, what percentage of the people will walk to work?

$$\begin{pmatrix} .7 & .4 \\ .3 & .6 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -.3 & .4 \\ .3 & -.4 \end{pmatrix}$$

$$\begin{pmatrix} -.3 & .4 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} -3 & 4 \\ 0 & 0 \end{pmatrix} \to \quad 3x_1 = 4x_2$$

$$x_1 = 4/3\, x_2$$

$$\begin{pmatrix} 4/3 \\ 1 \end{pmatrix} \text{ so } \begin{pmatrix} 4 \\ 3 \end{pmatrix} \text{ is a solution or } \begin{pmatrix} 4/7 \\ 3/7 \end{pmatrix}$$

So in long run 4/7 of people will walk or 57%