Grade: KEY

[16pts] 1. Let $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and suppose that $\det(A) = 3$. Compute the following determinants:

a) $\det(B)$, for $B = \begin{bmatrix} 2g & 2h & 2i \\ d & e & f \\ a & b & c \end{bmatrix}$

c) $\det(C)$, for $C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

$$\det(B) = \underbrace{(-1)}_{\text{interchange rows } 1+3} \underbrace{(2)}_{\text{multiply row 1 by 2}} \det(A)$$

$$= \boxed{(-2)(3) = -6}$$

$$\boxed{= (1)\det(A) = 3}$$

[10pts] 2. Compute $\det(A)$ for $A = \begin{pmatrix} 2 & -2 & 0 & 5 \\ -4 & 6 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

$$\det(A) = \det\begin{pmatrix} 2 & -2 \\ -4 & 6 \end{pmatrix} \det\begin{pmatrix} 2 & 3 \\ 1 & 1 \end{pmatrix}$$

$$= [(12) - 8][2 - 3]$$

$$\boxed{= (4)(-1) = -4}$$

[20pts] 3. Find the inverses of the following matrices:

(a) $A = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$

$$A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} 3 & -3 \\ -1 & 2 \end{pmatrix}$$

$$= \left(\frac{1}{6-3}\right)$$

$$= \frac{1}{3}\begin{pmatrix} 3 & -3 \\ -1 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} 1 & -1 \\ -1/3 & 2/3 \end{pmatrix}}$$

(b) $B = \begin{pmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{pmatrix}$

$$\begin{array}{cc} -2 & 3 \end{array} \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{array}\right)$$

$$3 \left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{array}\right) \frac{1}{2}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right) \begin{array}{cc} 2 & 2 \end{array}$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{array}\right)$$

$$\boxed{B^{-1} = \begin{pmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{pmatrix}}$$

[10pts] 4. Use the inverse you found in part (a) of the last problem to solve the system
$$\begin{aligned} 2x + 3y &= -2 \\ x + 3y &= 5 \end{aligned}$$

$$\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1/3 & 2/3 \end{pmatrix}\begin{pmatrix} -2 \\ 5 \end{pmatrix} = \begin{pmatrix} -7 \\ +2/3 + 10/3 \end{pmatrix}$$

$$= \begin{pmatrix} -7 \\ 4 \end{pmatrix}$$

$$\boxed{x = -7, \quad y = 4}$$

[12pt] 5. Find the LU factorization of $B = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 4 & 2 & 4 & 1 \\ 4 & 6 & 7 & 7 \end{pmatrix}$

$$\begin{pmatrix} 2 & 0 & 1 & 0 \\ 4 & 2 & 4 & 1 \\ 4 & 6 & 7 & 7 \end{pmatrix} \begin{matrix} \cdot 2 \\ -2 \\ \end{matrix} \rightarrow \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 \\ 0 & 6 & 5 & 7 \end{pmatrix} -3$$

$$\downarrow$$

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix}$$

$$U = \begin{pmatrix} 2 & 0 & 1 & 0 \\ 0 & 2 & 2 & 1 \\ 0 & 0 & -1 & 4 \end{pmatrix}$$

[12pts] 6. Use the following LU factorization of A to solve the matrix equation $A\mathrm{x} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

First Solve $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$$-1 \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{array}\right)$$

$$\downarrow$$

$$-2 \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & -1 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

Then solve

$$\frac{1}{2} \left(\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & -3 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & -3/2 \end{array}\right) -1$$

$$1 \left(\begin{array}{ccc|c} 2 & -1 & 0 & 1 \\ 0 & 1 & 0 & 5/2 \\ 0 & 0 & 1 & -3/2 \end{array}\right)$$

$$\frac{1}{2} \left(\begin{array}{ccc|c} 2 & 0 & 0 & 7/2 \\ 0 & 1 & 0 & 5/2 \\ 0 & 0 & 1 & -3/2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 7/4 \\ 0 & 1 & 0 & 5/2 \\ 0 & 0 & 1 & -3/2 \end{array}\right)$$

$$\boxed{\text{Ans} = \begin{pmatrix} 7/4 \\ 5/2 \\ -3/2 \end{pmatrix}}$$

[10pts] 7. Consider the following system of equations: $\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 - x_2 &= -1 \\ 3x_1 + 2x_3 &= 1 \end{aligned}$.

Use Cramer's rule to solve for x, given that $\det(A) = 9$, where A is the coefficient matrix of the above system.

$$b = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 3 & 0 & 2 \end{pmatrix}$$

$$A_1(b) = \begin{pmatrix} 0 & 2 & 1 \\ -1 & -1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$$

$$\det\left[\begin{pmatrix} 0 & 2 & 1 \\ -1 & -1 & 0 \\ 0 & -1 & 2 \end{pmatrix}\right] = (+1)\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} = 1(4+1) = 5$$

$$\boxed{\text{So } x_1 = \frac{5}{9}, \quad x_2 = \frac{3}{9} = \frac{1}{3}, \quad x_3 = \frac{-11}{9}}$$

[10pts] 8. Find the area of the parallelogram with vertices $(0,0)$, $(5,3)$, $(7,4)$, $(12,7)$. (It might help to draw the points carefully in the plane.)

$$\begin{bmatrix} 5 & 7 \\ 3 & 4 \end{bmatrix} = 20 - 21 = -1$$

$$|-1| = 1$$