1.3 VECTOR EQUATIONS

$\mathbb{R}^n$ is the collection of all lists (*ordered n-tuples*) of n real numbers. Example:

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}$$

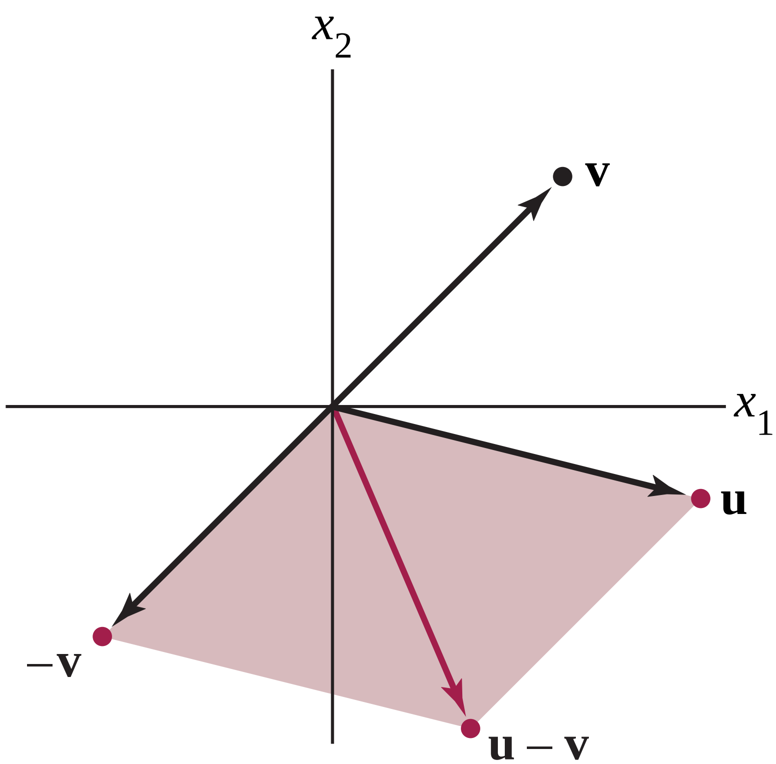
Algebraic Properties of $\mathbb{R}^n$

For all $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in $\mathbb{R}^n$ and all scalars c and d:

• $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$	commutative property
• $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$	associative property
• $\mathbf{u} + \mathbf{0} = \mathbf{0} + \mathbf{u} = \mathbf{u}$	zero vector
• $\mathbf{u} + (-\mathbf{u}) = -\mathbf{u} + \mathbf{u} = \mathbf{0}$	$-\mathbf{u}$ denotes $(-1)\mathbf{u}$
• $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$	distributive property
• $(c + d)\mathbf{u} + c\mathbf{u} + d\mathbf{u}$	
• $c(d\mathbf{u}) = (cd)\mathbf{u}$	
• $1\mathbf{u} = \mathbf{u}$	

Vector "Subtraction"

Write $\mathbf{u} - \mathbf{v}$ in place of $\mathbf{u} + (-1)\mathbf{v}$.



Linear Combinations

For vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$ in $\mathbb{R}^n$ and scalars $c_1, \ldots, c_p$, the vector

$$\mathbf{y} = c_1\mathbf{v}_1 + \cdots + c_p\mathbf{v}_p$$

is called a **linear combination** of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ using **weights** $c_1, \ldots, c_p$. Examples:

$$3.5\mathbf{v}_1 + 0\mathbf{v}_2 \quad (= 3.5\mathbf{v}_1), \qquad 0\mathbf{v}_1 + 0\mathbf{v}_2 \quad (= \mathbf{0})$$

EXAMPLE 4 Let $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Estimate the linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$ that generate $\mathbf{u}$ and $\mathbf{w}$.

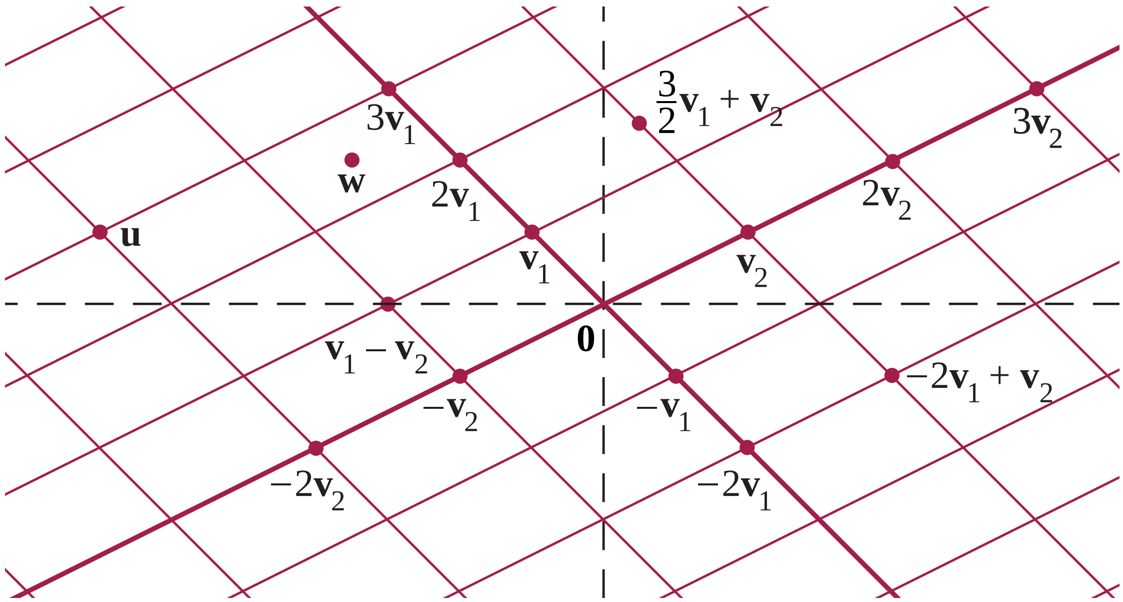
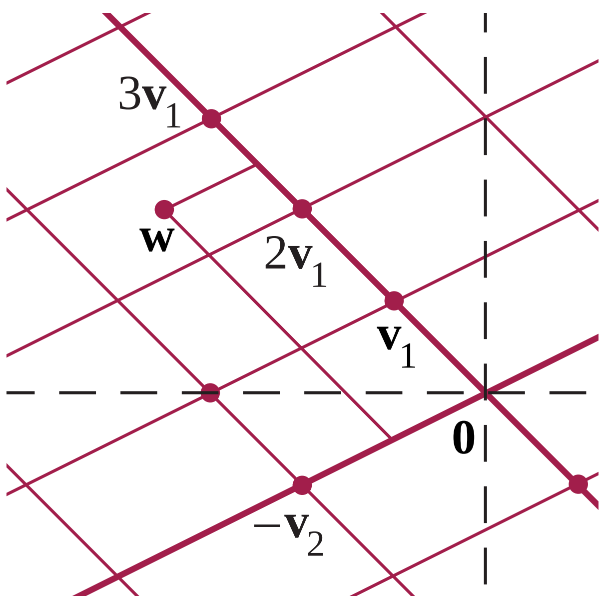


FIGURE 8 Linear combinations of $\mathbf{v}_1$ and $\mathbf{v}_2$.



EXAMPLE 5′ Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} -1 \\ 2 \\ 7 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$.

Determine whether $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. Do weights x_1 and x_2 exist such that

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b} \tag{1}$$

Solution

$$x_1 \underset{\displaystyle \uparrow \atop \displaystyle \mathbf{a}_1}{\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}} + x_2 \underset{\displaystyle \uparrow \atop \displaystyle \mathbf{a}_2}{\begin{bmatrix} -1 \\ -2 \\ 7 \end{bmatrix}} = \underset{\displaystyle \uparrow \atop \displaystyle \mathbf{b}}{\begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}}$$

Rewrite this vector equation:

$$\begin{bmatrix} x_1 \\ 0 \\ -3x_1 \end{bmatrix} + \begin{bmatrix} -x_2 \\ -2x_2 \\ 7x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

and

$$\begin{bmatrix} x_1 - x_2 \\ 0 - 2x_2 \\ -3x_1 + 7x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix} \tag{2}$$

$$x_1\mathbf{a}_1 \quad + \quad x_2\mathbf{a}_2 \quad = \quad \mathbf{b}$$

$$x_1\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} + x_2\begin{bmatrix} -1 \\ -2 \\ 7 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 - x_2 \\ 0 - 2x_2 \\ -3x_1 + 7x_2 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \\ 1 \end{bmatrix}$$

$$\begin{aligned} x_1 - x_2 &= -3 \\ 0 - 2x_2 &= 4 \\ -3x_1 + 7x_2 &= 1 \end{aligned}$$

Solve this system by row reducing the augmented matrix:

$$\begin{bmatrix} 1 & -1 & -3 \\ 0 & -2 & 4 \\ -3 & 7 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -3 \\ 0 & -2 & 4 \\ 0 & 4 & -8 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -3 \\ 0 & -2 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

The system is consistent, so $\mathbf{b}$ is a linear combination of $\mathbf{a}_1$ and $\mathbf{a}_2$. To find the weights, complete the row reduction, and obtain $x_1 = -5$, $x_2 = -2$. Thus

$$-5\mathbf{a}_1 - 2\mathbf{a}_2 = \mathbf{b}$$

Summary:

To study the equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$, consider:

$$\begin{bmatrix} 1 & -1 & -3 \\ 0 & -2 & 4 \\ -3 & 7 & 1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b} \end{bmatrix}$$

$$\begin{matrix} \uparrow & \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b} \end{matrix}$$

A vector equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n = \mathbf{b}$$

has the same solution set as the linear system whose augmented matrix is

$$\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n & \mathbf{b} \end{bmatrix} \qquad (*)$$

In particular, $\mathbf{b}$ *can be generated by a linear combination of* $\mathbf{a}_1, \ldots, \mathbf{a}_n$ *if and only if the linear system corresponding to (*) has a solution.*

Definition. If $\mathbf{v}_1, \ldots, \mathbf{v}_p$ are vectors in $\mathbb{R}^n$, then the set of all possible linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is denoted by

$$\text{Span}\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$$

Vectors of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

Note: Every scalar multiple of $\mathbf{v}_1$ (for example) is in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, because

$$c\mathbf{v}_1 = c\mathbf{v}_1 + 0\mathbf{v}_2 + \cdots + 0\mathbf{v}_p$$

The zero vector is always in Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

The only vectors in Span$\{\mathbf{v}_1\}$ are multiples of $\mathbf{v}_1$.

A Geometric Description of Span{**v**}

If **v** in $\mathbb{R}^3$ is nonzero, then Span{**v**} is the set of points on the line in $\mathbb{R}^3$ through **v** and the origin.

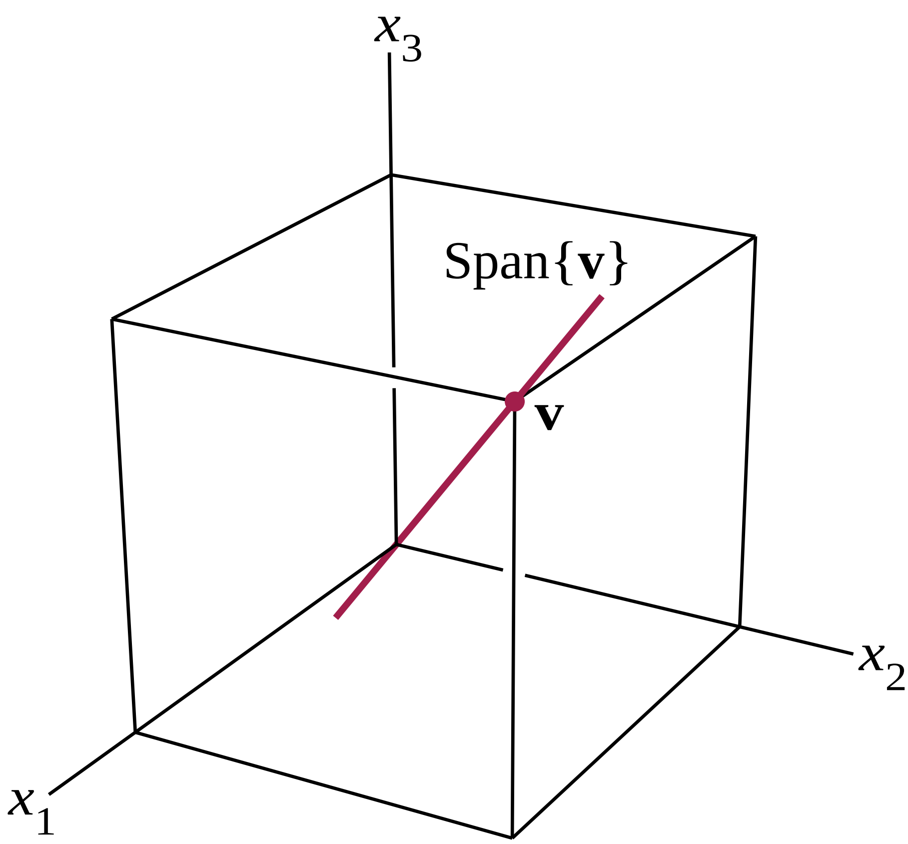


FIGURE 10 Span{**v**} as a line through the origin.

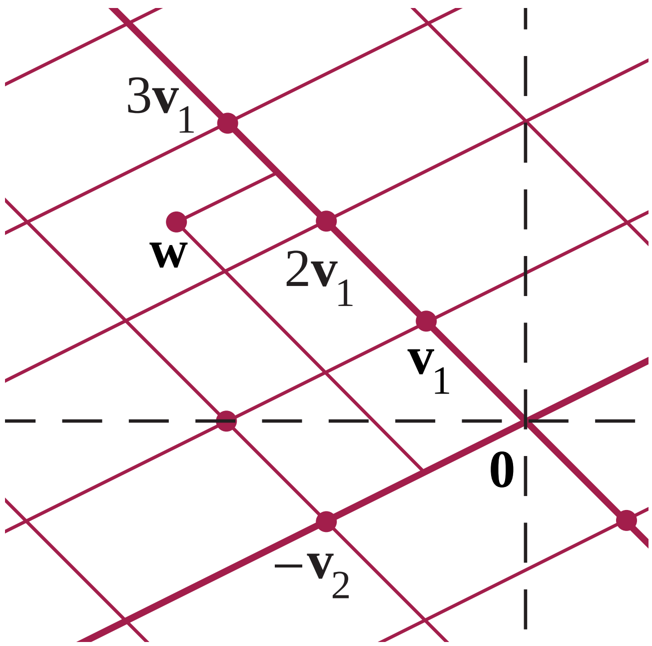


FIGURE 9

EXAMPLE 6′ Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} -1 \\ -2 \\ 7 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} -3 \\ 4 \\ 2 \end{bmatrix}$.
(Same $\mathbf{a}_1$, $\mathbf{a}_2$ as in Example 5′.) Determine whether $\mathbf{b}$ is in the plane spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$.

Solution Does $x_1\mathbf{a}_1 + x_2\mathbf{x}_2 = \mathbf{b}$ have a solution?
Row reduce $\begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{b} \end{bmatrix}$:

$$\begin{bmatrix} 1 & -1 & -3 \\ 0 & -2 & 4 \\ -3 & 7 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -3 \\ 0 & -2 & 4 \\ 0 & 4 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -3 \\ 0 & -2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$$

The vector equation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{b}$ has no solution.
So $\mathbf{b}$ is *not* in Span$\{\mathbf{a}_1, \mathbf{a}_2\}$.