1.1 SYSTEMS OF LINEAR EQUATIONS

A **linear equation**:

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$$

Examples:

$$4x_1 - 5x_2 + 2 = x_1 \quad \text{and} \quad x_2 = 2(\sqrt{6} - x_1) + x_3$$

Not linear:

$$4x_1 - 5x_2 = x_1x_2 \quad \text{and} \quad x_2 = 2\sqrt{x_1} - 6$$

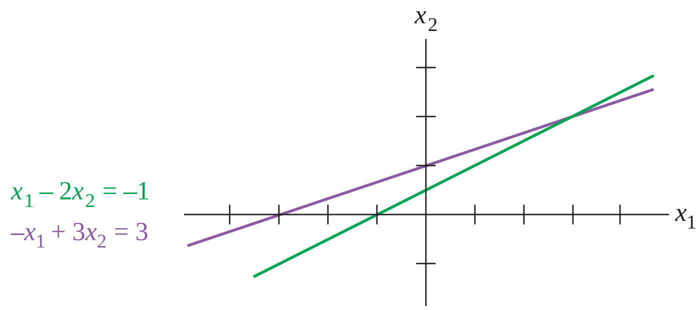
A **system of linear equations** (or a **linear system**):

A collection of one or more linear equations involving the same set of variables, say, $x_1, \ldots, x_n$.

A **solution** of the system:

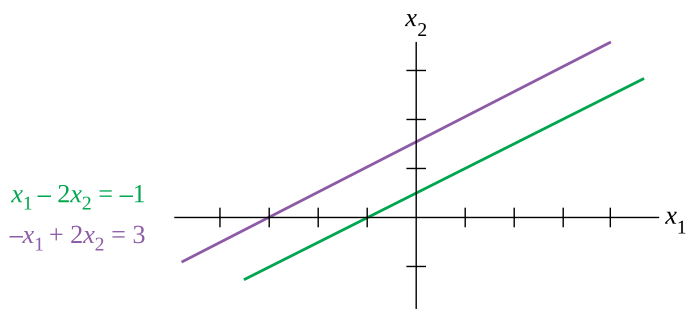
A list $(s_1, s_2, \ldots, s_n)$ of numbers that makes each equation in the system a true statement when the values $s_1, \ldots, s_n$ are substituted for $x_1, \ldots, x_n$, respectively.

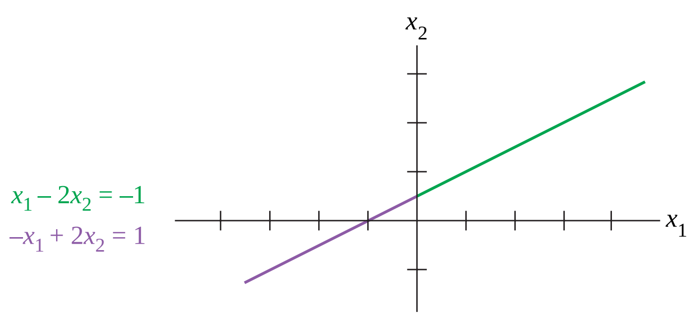
EXAMPLE Two equations in two variables:



A solution is a pair (x_1, x_2) that lies on both lines.

Two other possibilities:





Basic Fact: *A system of linear equations has either*

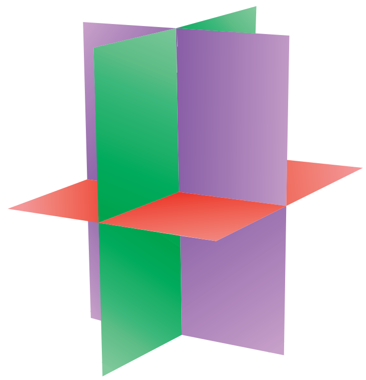
(i) exactly one solution; or **consistent**

(ii) infinitely many solutions; or **consistent**

(iii) no solution. **inconsistent**

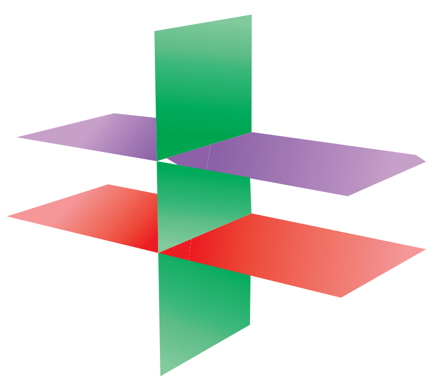
EXAMPLE Three equations in three variables. Each equation determines a plane in space.

i) The planes intersect in one point:



ii) The planes intersect in a line:

iii) There is no point common to all three planes:



The **solution set**:

The set of all possible solutions of the system.

Equivalent systems:

Two linear systems with the same solution set.

STRATEGY FOR SOLVING A SYSTEM:

Replace one system with an equivalent system that is easier to solve.

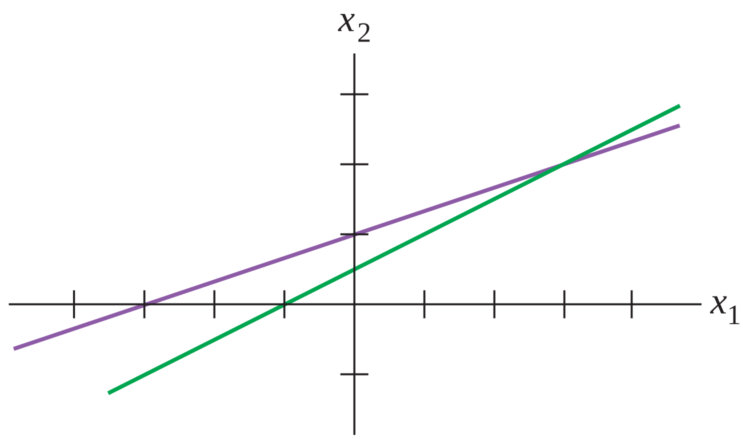
EXAMPLE

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \quad \rightarrow \quad \begin{aligned} x_1 - 2x_2 &= -1 \\ x_2 &= 2 \end{aligned}$$

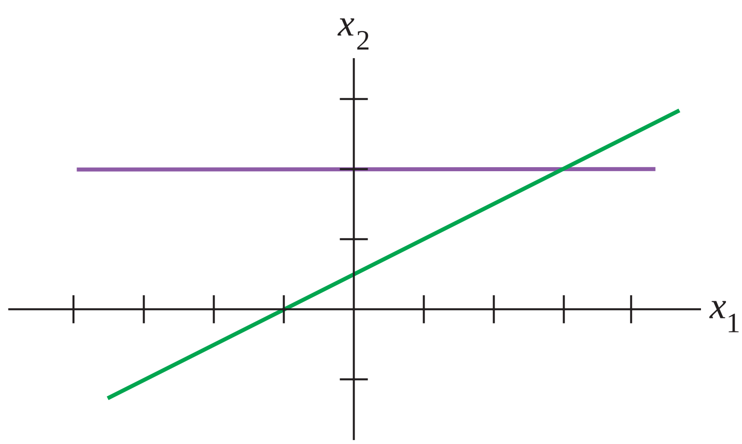
$$\rightarrow \quad \begin{aligned} x_1 &= 3 \\ x_2 &= 2 \end{aligned}$$

EXAMPLE

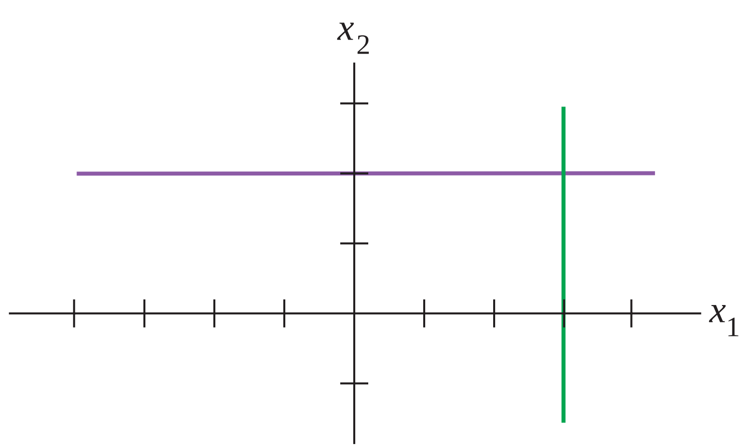
$$x_1 - 2x_2 = -1$$
$$-x_1 + 3x_2 = 3$$



$$x_1 - 2x_2 = -1$$
$$x_2 = 2$$



$$x_1 = 3$$
$$x_2 = 2$$



MATRIX NOTATION

coefficient matrix:

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}$$

augmented matrix:

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ -x_1 + 3x_2 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & -3 \\ -1 & 3 & 3 \end{bmatrix}$$

$$\begin{aligned} x_1 - 2x_2 &= -1 \\ x_2 &= 2 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\begin{aligned} x_1 \qquad &= 3 \\ x_2 &= 2 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix}$$

Elementary row operations:

1. *(Replacement) Add to one row a multiple of another row.*
2. *(Interchange) Interchange two rows*
3. *(Scaling) Multiply all entries in a row by a nonzero constant.*

Fact about Row Equivalence: *If the augmented matrices of two linear systems are now equivalent, then the two systems have the same solution set.*

EXAMPLE 1

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{bmatrix}$$

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{aligned} x_1 - 2x_2 &= -3 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

$$\begin{aligned} x_1 &= 29 \\ x_2 &= 16 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Solution is (29, 16, 3)

Check:

Is (29, 16, 3) a solution of the original system?

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

Substitute and compute:

$$\begin{aligned} (29) - 2(16) + (3) &= 29 - 32 + 3 = 0 \\ 2(16) - 8(3) &= 32 - 24 \phantom{{}+27} = 8 \\ -4(29) + 5(16) + 9(3) &= -116 + 80 + 27 = -9 \end{aligned}$$

TWO FUNDAMENTAL QUESTIONS

(1) Is the system consistent; that is, does a solution ***exist****?*

(2) If a solution exists, is it the only one; that is, is the solution ***unique****?*

EXAMPLE 2 Is this system consistent?

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

Solution In Example 1 we row reduced this system to the "triangular" form:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

Now we can see that a solution exists and it is unique. (Why?)

EXAMPLE 3′ Is this system consistent?

$$\begin{aligned} 3x_2 - 6x_3 &= 8 \\ x_1 - 2x_2 + 3x_3 &= -1 \\ 5x_1 - 7x_2 + 9x_3 &= 0 \end{aligned} \qquad \begin{bmatrix} 0 & 3 & -6 & 8 \\ 1 & -2 & 3 & -1 \\ 5 & -7 & 9 & 0 \end{bmatrix}$$

Solution Row operations on the augmented matrix:

$$\begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 5 & -7 & 9 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 3 & -6 & 5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 3 & -1 \\ 0 & 3 & -6 & 8 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

To interpret this "triangular form" go back to equation notation:

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= -1 \\ 3x_2 - 6x_3 &= 8 \\ 0 &= -3 \qquad \leftarrow \textit{Never true!} \end{aligned}$$

EXAMPLE For what values of h will the following system be consistent?

$$\begin{aligned} 3x_1 - 9x_2 &= 4 \\ -2x_1 + 6x_2 &= h \end{aligned}$$

Solution Reduce the system to triangular form.

Add 2/3 times row 1 to row 2:

$$\begin{aligned} 3x_1 - 9x_2 &= 4 \\ 0x_1 + 0x_2 &= h + 8/3 \end{aligned}$$

only true if $h + 8/3 = 0$

The system is consistent precisely when $h = -8/3$.