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Research Statement
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My research area is in the area of combinatorics with a focus on discrete geometry. My work reflects the interplay of graph theory with discrete geometry. I have had the good fortune to gain research experience in a number of disciplines, not only as a graduate student but also as an undergraduate at two Research Experience for Undergraduates (REUs).

1 Convex Geometries

My motivation is inspired by a problem of Erdős and Szekeres: for any $n \geq 3$, to determine the smallest positive integer $N(n)$ such that any set of at least $N(n)$ points in general position (no three on a line) in the plane contains n points that are the vertices of a convex n-gon. Morris and Soltan [13] surveyed known results related to this problem. It has been established that $N(n)$ exists and $N(n) > 2^{n-2}$ and conjectured that $N(n) = 2^{n-2} + 1$ by Erdős and Szekeres ([7],[8]). It is currently known that $N(n) \leq \binom{2n-5}{n-2} + 1$, due to Tóth and Valtr ([17]).

Let X be a finite set and $\mathscr{L}$ be a collection of subsets of X with the properties: $\emptyset \in \mathscr{L}$, $X \in \mathscr{L}$, and $A \cap B \in \mathscr{L}$ whenever $A, B \in \mathscr{L}$. Then $\mathscr{L}$ is called an *alignment* of X. Following the example of Edelman and Jamison ([5]), $\mathscr{L}$ is also viewed as a *closure operator*. For any subset A of X, define the closure of A, $\mathscr{L}(A)$, be the intersection of all $C \in \mathscr{L}$ such that $A \subseteq C$. The subsets in $\mathscr{L}$ or equivalently those subsets of X of the form $\mathscr{L}(A)$ for some subset A of X are called *closed* or *convex*. The closure operator $\mathscr{L}$ is *anti-exchange* if given any set $C \in \mathscr{L}$ and two distinct points p and q in X, neither in C, then $q \in \mathscr{L}(C \cup p)$ implies that $p \notin \mathscr{L}(C \cup q)$.

Definition 1.1. Let X be a finite set. A pair $(X, \mathscr{L})$ is a *convex geometry* if:

1) $\mathscr{L}$ is an alignment of X,

2) $\mathscr{L}$ is anti-exchange.

Edelman and Jamison [5] presented several equivalent definitions of convex geometries.

For an alignment $(X, \mathscr{L})$ denote by $L_{\mathscr{L}} = (\mathscr{L}, \subseteq)$ the partial order on $\mathscr{L}$ by containment. This partial order is a lattice where $A \wedge B = A \cap B$ and $A \vee B = \mathscr{L}(A \cup B)$. A closed subset of a convex geometry, $A \in \mathscr{L}$, is a *copoint* if there is exactly one $B \in \mathscr{L}$ such that $|B - A| = 1$. Copoints are the meet-irreducible elements of $L_{\mathscr{L}}$. The unique element in $B - A$ is denoted $\alpha(A)$. It is said that the copoint A is *attached* to $\alpha(A)$. The set of copoints partially ordered by inclusion is denoted $M(X)$. The set $B \subseteq X$ is *independent* if for all $p \in B$, $p \notin \mathscr{L}(B - p)$. The size of the largest independent set is denoted by $b(L_{\mathscr{L}})$.

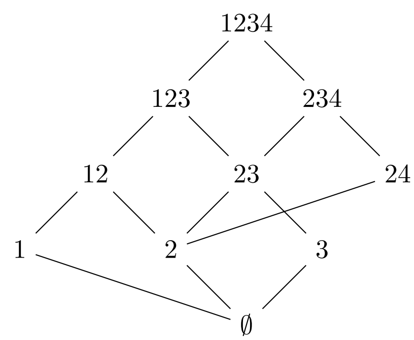

Figure 1: The lattice of closed sets for a non-atomic convex geometry

Definition 1.2. Let X be a finite set of points in $\mathbb{R}^2$. The convex geometry $(X, \mathscr{L})$ *realized* by X is defined by $\mathscr{L}(A) = \text{conv}(A) \cap X$ for all $A \subseteq X$.

The order dimension of a poset, $P = (X, \leq)$ is the least positive integer t for which there exists a family $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of P so that $P = \cap\mathcal{R}$. Any family of linear extensions $\mathcal{R}$ such that $\cap\mathcal{R} = P$ is called a *realizer* of P. For a poset P denote the order dimension of P by $\dim(P)$. For any convex geometry, $\dim(L_{\mathscr{L}}) \geq b(L_{\mathscr{L}})$ with strict inequality possible ([6]). Figure 1 is the lattice of closed sets of such an example with $\dim(L_{\mathscr{L}}) = 3$ and $b(L_{\mathscr{L}}) = 2$.

Given two disjoint planar point sets L and M, a *composition* of L and M is defined to be a point set of L together with a translation of M in which

1) every point of M has greater first coordinate than the first coordinates of points of L,

2) the slope of any line connecting a point of L to a point of M is greater than the slope of any line connecting two points of L or two points of M.

Erdős and Szekeres [8] describe a construction of large point sets without large subsets in convex position. I describe these point sets with the notation of [12]. For all positive integers, k, I first define $ES(0, k)$ and $ES(k, 0)$ to be singletons. For $i \geq 1, j \geq 1$ define $ES(i, j)$ to be a composition of $ES(i-1, j)$ and $ES(i, j-1)$.

The extended Erdős-Szekeres point set $XES(k)$ is a composition of $ES(0, k)$, $ES(1, k-1)$, $\cdots$, $ES(k, 0)$ where the compositions are performed in order from left to right. The number of points in $XES(k)$ is 2^k. The size of the largest independent set in $XES(k)$ is $k+1$ ([12]). I was able to show that this is also the order dimension of the lattice of closed sets for $XES(k)$ ([1]). Moreover, any planar point set in general position with more points, must have a larger order dimension.

Theorem 1.3. *([1]) If X is a planar point set in general position and* $\dim(X) = k$*, then* $|X| \leq 2^{k-1}$.

Theorem 1.3 was proven by coloring a graph of Morris ([12]) and relating it to the graph studied by Felsner and Trotter([9]). Morris' graph, $\mathcal{G}(X, \mathscr{L})$, has vertex set equal to the set of copoints of the convex geometry and there is an edge between two copoints A and B if and only if $\alpha(A) \in B$ and $\alpha(B) \in A$.

Felsner and Trotter were concerned with posets while Morris' work was on convex geometries realized by planar point sets in general position. Taking the work of these authors together, one can ask many interesting and deep questions.

Question 1.4. $\chi(\mathcal{G}(X, \mathscr{L})) = \dim(L_{\mathscr{L}})$ *for all examples studied. Is this true in general or is it possible to find a counterexample?*

Question 1.5. *Is there some fixed constant c such that $c \geq \frac{\chi(\mathcal{G}(X,\mathscr{L}))}{\omega(\mathcal{G}(X,\mathscr{L}))}$? If $c < 2$, then it would improve the bound of Tóth and Valtr.*

Question 1.5 has been partially solved in joint work with Walter Morris ([2]). For abstract convex geometries, there is no such constant. However, for convex geometries realized by planar point sets in general position, this is still open.

While investigating Question 1.5, a new related problem to the Erdős-Szekeres problem arose. It starts with the following theorem:

Theorem 1.6. *([2]) Let $(X, \mathscr{L})$ be a convex geometry with every two element subset closed. If $|X| = 5$, then $\chi(\mathcal{G}(X, \mathscr{L})) \geq 4$.*

The *Carathéodory number* of a convex geometry $(X, \mathscr{L})$ is the least positive integer c such that $\mathscr{L}(Y) = \cup\{\mathscr{L}(Z) : Z \subseteq Y, |Z| \leq c\}$ for any $Y \subseteq X$. A set $Y \subseteq X$ is said to be in *nice position* if any c points of Y are convexly independent. It is a simple exercise to show that every subset of size $c-1$ or less of Y is a closed set. Such a convex geometry is said to be $(c-1)-free$. In light of Theorem 1.6, and extending a lemma of Morris and Soltan ([13]) there is also the following result.

Theorem 1.7. *([2]) Let $(X, \mathscr{L})$ be a convex geometry with Carathéodory number c and $Y \subseteq X$ a subset in nice position such that $|Y| = c + 2$. Then $\chi(\mathcal{G}(Y, \mathscr{L}|_Y)) \geq c + 1$.*

So, we pose the following problem:

Problem 1.8. *For any integer $n \geq d \geq 2$, determine the smallest positive integer $K_d(n)$ such that any d-free convex geometry of $K_d(n)$ points requires that $\chi(\mathcal{G}(X, \mathscr{L})) \geq n$.*

Problem 1.8 can be completely solved when restricted to those convex geometries realized by planar point sets, $K_2^p(n) = 2^{n-2} + 1$. Another case where the this problem has been solved is when $d = 2$, this problem closely resembles the order dimension of the K_n ([10]). A family of subsets of $[t]$ is called *intersecting* if $A \cap B \neq \emptyset$ whenever $A, B \in [t]$. An intersecting family of subsets is *maximal* if it is contained in no other intersecting family. Let $\gamma(n)$ denote the number of maximal intersecting families of subsets of an n-element set. It is a well known result that $\gamma(n)$ is at least $2^{\binom{n-1}{\lfloor (n-1)/2 \rfloor}}$ ([15]).

Theorem 1.9. *([2])* $K_2(n) = \gamma(n)$

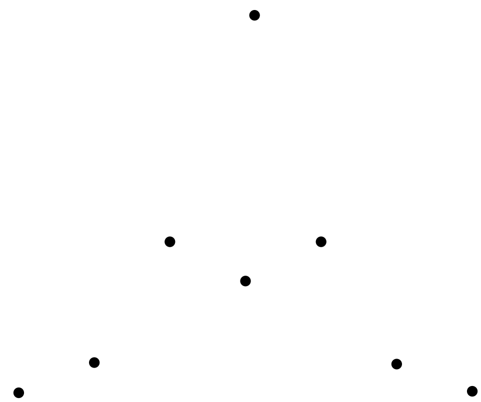

Figure 2: An eight point set in general position

If $d > 2$, then a d-free convex geometry is necessarily 2-free as well. This gives us the general corollary.

Corollary 1.10. *([2]) If $d \geq 2$, then $K_d(n)$ is finite and $K_d(n) \leq 2^{\binom{n-1}{\lfloor (n-1)/2 \rfloor}}$*

It is also possible to improve the results of Theorem 1.3 by studying the linear relaxation of chromatic number as defined by Scheinerman and Ullman [14], the fractional chromatic number, denoted $\chi_f(G)$. This is important because there are examples where $\chi(\mathcal{G}(X, \mathscr{L})) > \omega(\mathcal{G}(X, \mathscr{L}))$. The graph of the point set in Figure 2 has chromatic number 5, fractional chromatic number 4.5, and clique number 4.

Question 1.11. *Let X be a planar point set in general position. If $\chi_f(\mathcal{G}(X, \mathscr{L})) = k$, is then $|X| \leq 2^{k-1}$?*

Beyond the problem of Erdős and Szekeres, there are many other problems in extremal combinatorial geometry that are the subject of wide interest. The book of Brass, Moser, and Pach ([3]) is filled with such problems with many references. Among these are the crossing number of a graph, the number of incidences between n points and m lines, number of k-edges and k-sets, Sylvester's problem of ordinary lines, and generalizations to higher dimensions.

Question 1.12. *Is it possible to study other problems in combinatorial geometry by use of parameters of the graphs $\mathcal{G}(X, \mathscr{L})$ for appropriate choices of the convex geometry $(X, \mathscr{L})$? If not a convex geometry, an appropriate poset P and the graph of Felsner and Trotter?*

Székely ([16]) has used graphs on a number of occasions to improve known results in discrete geometry.

2 Undergraduate Research

Many problems in discrete geometry are understandable and relatable to undergraduate students without an extensive or wide background. As with many combinatorial problems, the techniques used to create solutions or counter-examples are often elegant and can be thought of by anyone with a fresh approach.

Indeed, recently, some REU students ([11]) were able to construct a counterexample to a conjecture of Székely and de Caen ([4]).

There are many questions about the graph $\mathcal{G}(X, \mathscr{L})$ that I believe can be understood and worked on by undergraduates. Since this graph is new and mostly understudied, there are many fundamental properties that undergraduates would be able to understand. These properties may lead toward eventually shrinking the gap between the results for the chromatic number of $\mathcal{G}(X, \mathscr{L})$ and the clique number of $\mathcal{G}(X, \mathscr{L})$.

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