# CHROMATIC NUMBERS OF COPOINT GRAPHS OF CONVEX GEOMETRIES

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Abstract. We study the copoint graph of a convex geometry. We give a family of copoint graphs for which the ratio of the chromatic number to the clique number can be arbitrarily large. For any natural numbers $1 < d < k$, we study the existence of a number $K_d(k)$ so that the chromatic number of the copoint graph of a convex geometry on a set of at least $K_d(k)$ elements, with every $d$-element subset closed, has chromatic number at least $k$. Our results are analogues of results of Erdős and Szekeres for convex geometries realizable by point sets in $R^m$, where cliques in the copoint graph correspond to subsets of points in convex position.

## 1. Introduction

Let $X$ be a finite set. An *alignment* $\mathscr{L}$ is a collection of subsets of $X$ such that $\emptyset \in \mathscr{L}, X \in \mathscr{L}$, and if $A, B \in \mathscr{L}$ then $A \cap B \in \mathscr{L}$. A set $C \subseteq X$ is *closed* or *convex* if $C \in \mathscr{L}$. Following Edelman and Jamison [4], we also view $\mathscr{L}$ as a closure operator on the subsets of $X$, where $\mathscr{L}(A) = \bigcap\{C : C \text{ is closed and } A \subseteq C\}$. The closure operator $\mathscr{L}$ is *anti-exchange* if for any $x, y \notin \mathscr{L}(C), x \in \mathscr{L}(C \cup y)$, then $y \notin \mathscr{L}(C \cup x)$. Equivalently, for any closed set $C$, with $C \neq X$, there is at least one closed set of the form $C \cup p$ for $p \notin C$. A pair $(X, \mathscr{L})$ where $\mathscr{L}$ is an anti-exchange closure operator is called a *convex geometry*. The closed sets of a convex geometry $(X, \mathscr{L})$ can be partially ordered by inclusion to form a lattice, $L_{\mathscr{L}}$. A subset $A \subseteq X$ is *convexly independent* or *independent* if for all $p \in A, p \notin \mathscr{L}(A - p)$.

A set $C \in \mathscr{L}$ is a *copoint* if it is maximal in $X - p$ for some $p \in X$. If $C$ is a copoint, there is exactly one set in $\mathscr{L}$ of the form $C \cup p$ for $p \notin C$. The unique $p$ is denoted $\alpha(C)$, and we say that the copoint $C$ is *attached* to $\alpha(C)$. We will sometimes refer to a copoint $C$ by the pair $(\alpha(C), C)$. The *copoint graph* of $(X, \mathscr{L})$, $\mathcal{G}(X, \mathscr{L})$, has as its vertex set the set of copoints of $(X, \mathscr{L})$, with copoints $C$ and $D$ adjacent if and only if $\alpha(C) \in D$ and $\alpha(D) \in C$. The definition of independent sets shows that a set $A \subseteq X$ is independent in $(X, \mathscr{L})$, if and

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only if there is a clique in $\mathcal{G}(X, \mathscr{L})$ of copoints attached to the elements of $A$. Thus the clique number of $\mathcal{G}(X, \mathscr{L})$ equals the size of the largest independent set of $(X, \mathscr{L})$.

If $X$ is a set of points in $\mathbb{R}^d$, and $\mathscr{L} = \{C \subseteq X : X \cap conv(C) = C\}$, then $(X, \mathscr{L})$ is a convex geometry, called the convex geometry realized by $X$. One can show that if the points of a set $X$ are in general position in $\mathbb{R}^d$ and a set $A \subseteq X$ is the vertex set of a convex polytope, then $A$ is independent in $(X, \mathscr{L})$. For point sets $X$ in general position in $\mathbb{R}^2$, there is a famous conjecture of Erdős and Szekeres [6] that $X$ contains the vertex set of a convex $n$-gon whenever $|X| > 2^{n-2}$. Morris [13] proved that for a point set $X$ in general position in $\mathbb{R}^2$, the *chromatic* number of $\mathcal{G}(X, \mathscr{L})$ is at least $n$ whenever $|X| > 2^{n-2}$. This result highlights the need to understand the relationship between the chromatic number and clique number of copoint graphs. We will present several results involving the clique and chromatic number of copoint graphs for general convex geometries. One should keep in mind, however, that convex geometries realized by point sets in $\mathbb{R}^d$ form a small subset of the set of all convex geometries.

In Section 2 we answer a question posed by Beagley ([2]), giving a family of convex geometries $([n], \mathscr{L}_{m,n})$, for positive integers $m < n$, for which $\omega(\mathcal{G}(X, \mathscr{L})) = m + 1$ and $\chi(\mathcal{G}(X, \mathscr{L})) \geq \lceil log_2(n + 1)\rceil$. This shows that the ratio of the chromatic number of $\mathcal{G}(X, \mathscr{L})$ to the clique number of $\mathcal{G}(X, \mathscr{L})$ can be arbitrarily large. The convex geometry $([n], \mathscr{L}_{m,n})$ will have the property that it is *m-free*, i.e. $\mathscr{L}$ will contain every $m$-element subset of $[n]$. This is a property that is satisfied by convex geometries realized by point sets in general position in $\mathbb{R}^m$.

In Section 3 we investigate the effect that the $d$-free property alone will have on the chromatic number of $\mathcal{G}(X, \mathscr{L})$. The main results we prove are that if $1 < d < k$ there exists a number $K_d(k)$ so that any $d$-free convex geometry on a set of size at least $K_d(k)$ will have the chromatic number of its copoint graph at least $k$. We also show that $K_d(d + 2) = d + 3$ for all $d > 1$, analogous to a result of Esther Klein that every set of 5 planar points in general position contains the vertex set of a convex 4-gon.

To close this introductory section, we give the smallest set of points in the plane in general position for which $\omega(\mathcal{G}(X, \mathscr{L}))$ and $\chi(\mathcal{G}(X, \mathscr{L}))$ differ.

Of the 16 order types of 6 planar points in general position [1], there is only one with this property. It is given in Figure 1. The copoints are shown to the right of the point set, in the

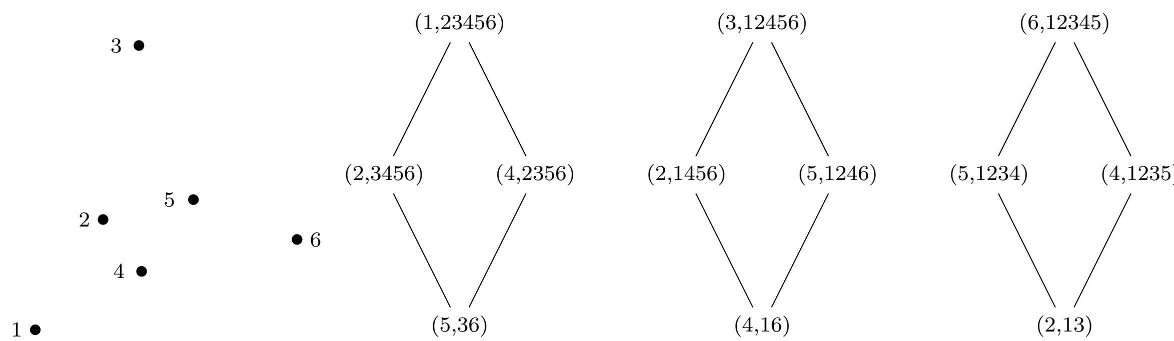


FIGURE 1. A six point set and its poset of copoints

form $(\alpha(C), C)$ where $\alpha(C)$ is the point to which copoint $C$ is attached. The copoints are partially ordered by set containment. The subgraph of $\mathcal{G}(X, \mathscr{L})$ induced by the copoints of $(X, \mathscr{L})$ of size bigger than 3 form the complement of a 9-cycle. This graph has chromatic number 5 and clique number 4.

## 2. Construction of A Convex Geometry

Beagley [2] asked the following question: Is $\frac{\chi(\mathcal{G}(X,\mathscr{L}))}{\omega(\mathcal{G}(X,\mathscr{L}))} \leq c$ for some constant $c$? We construct a family of convex geometries indexed by integers $m, n$ with clique number of $m+1$ and chromatic number at least $\lceil log_2(n+1) \rceil$.

Let $n$ be a positive integer and $\{1, 2, \ldots, n\} = [n]$. When $n = 0$, then $[n] = \emptyset$. Let $m$ be a positive integer, $m < n$, and define $\mathscr{L}_{m,n} = \{([i] \cup J) | 0 \leq i \leq n, J \subseteq \{i+2, \ldots, n\}, |J| \leq m\}$.

**Proposition 2.1.** *For $n, m$ positive integers with $m < n$, the pair $([n], \mathscr{L}_{m,n})$ is an $m$-free convex geometry.*

*Proof.* It is easy to see that $\mathscr{L}_{m,n}$ is closed under intersection and $\emptyset, [n] \in \mathscr{L}_{m,n}$. Let $C$ be in $\mathscr{L}_{m,n}, C \neq [n]$. If $C = [i] \cup J$ with $0 \leq i \leq n, J \subseteq \{i+2, \ldots, n\}, |J| \leq m$, then $C \cup \{i+1\} \in \mathscr{L}_{m,n}$, so $([n], \mathscr{L}_{m,n})$ is a convex geometry. To see that $([n], \mathscr{L}_{m,n})$ is $m$-free, note that if $|J| \leq m$ and $i$ is the smallest element of $[n] \backslash J$, then $J = [i-1] \cup J'$ where $|J'| \leq m$. $\square$

For each $i \in \{1, 2, \ldots, n-m\}$, define $A_i = \{[i-1] \cup J | J \subseteq \{i+1, i+2, \ldots, n\}, |J| = m\}$ and for each $i \in \{n-m+1, n-m+2, \ldots, n\}$ let $A_i = \{[i-1] \cup \{i+1, i+2, \ldots, n\}\}$.

**Proposition 2.2.** *For $i = 1, 2, \ldots, n$, $A_i$ is the set of copoints of $([n], \mathscr{L}_{m,n})$ attached to $i$.*

*Proof.* If $C = [i-1] \cup J \neq [n]$ for $1 \leq i \leq n$, $J \subseteq \{i+1, \ldots, n\}$, $|J| \leq m$, then $C \cup \{i\}$ is in $\mathscr{L}_{m,n}$. If $C$ is not in $A_i$, then there is an element $p \neq i$ of $[n]$ such that $C \cup \{p\}$ is in $\mathscr{L}_{m,n}$. □

The size of the maximum clique in $\mathcal{G}([n], \mathscr{L}_{m,n})$ can be found using the size of the largest independent set.

**Lemma 2.3.** *The clique number of* $\mathcal{G}([n], \mathscr{L}_{m,n})$ *is* $m+1$.

*Proof.* Let $C \in \mathscr{L}_{m,n}$. If $|C| \leq m$, then $\mathscr{L}_{m,n}(C) = C$. So, let $|C| > m$. We can write $C = [i] \cup J$ where $0 \leq i \leq n-m$, $J \subseteq \{i+1, \ldots, n\}$, $|J| = m$. Thus, $C = \mathscr{L}_{m,n}(\{i\} \cup J)$ and $|\{i\} \cup J| = m+1$. Further, $\mathcal{G}([n], \mathscr{L}_{m,n})$ contains a $m+1$-clique consisting of the copoints of the form $[n] \backslash \{i\}$ for $i = n-m, \ldots, n$. Since every closed set $C$ can be written as the closure of at most $m+1$ elements of $[n]$, there is no independent set of size $m+2$ and $\omega(\mathcal{G}([n], \mathscr{L}_{m,n})) = m+1$. □

To bound the chromatic number of $\mathcal{G}([n], \mathscr{L}_{m,n})$ we make note of the following property of the set $A_i$.

**Proposition 2.4.** *Suppose that* $B \subseteq [n]$, $|B| \leq m$, *and* $i < b$ *for all* $b \in B$. *Then there exists* $C \in A_i$ *so that* $C$ *contains every element of* $B$.

*Proof.* Choose a copoint $C = [i-1] \cup J$ in $A_i$ with $B \subseteq J$, and the result is immediate. □

**Corollary 2.5.** *Suppose that* $B \subseteq [n]$, $|B| \leq m$, *and that* $i < b$ *for all* $b \in B$. *Then there exists* $C \in A_i$ *so that* $C$ *is adjacent in* $\mathcal{G}([n], \mathscr{L}_{m,n})$ *to every copoint* $D$ *in* $\bigcup_{b \in B} A_b$.

*Proof.* By the previous proposition, $b \in C$ for every $b \in B$, and because $b > i$, we have $i \in [b-1] \subseteq D$ for all $D \in A_b$. □

We shall answer Beagley's question, using $([n], \mathscr{L}_{m,n})$ to show that the ratio $\frac{\chi(\mathcal{G}(X \mathscr{L}))}{\omega(\mathcal{G}(X, \mathscr{L}))}$ is bounded by no constant $c$.

**Theorem 2.6.** *The convex geometry* $([n], \mathscr{L}_{m,n})$ *has* $\omega(\mathcal{G}([n], \mathscr{L}_{m,n})) = m+1$ *and* $\lceil log_2(n+1) \rceil \leq \chi(\mathcal{G}([n], \mathscr{L}_{m,n}))$.

*Proof.* $\omega(\mathcal{G}([n], \mathscr{L}_{m,n})) = m + 1$ by Lemma 2.3.

For any proper coloring of $\mathcal{G}([n], \mathscr{L}_{m,n})$ with $c$ colors, let $S_i$ be the set of colors used to color the copoints of $A_i$, $i = 1, 2, \ldots, n$. For $1 \leq i < j \leq n$, the fact that there is a copoint of $A_i$ adjacent to every copoint of $A_j$ means that the $S_i$ are distinct and nonempty. Therefore, $n \leq 2^c - 1$, and any proper coloring of $\mathcal{G}([n], \mathscr{L}_{m,n})$ requires at least $\lceil log_2(n+1) \rceil$ colors. $\square$

The graph for the convex geometry $([n], \mathscr{L}_{m,n})$ has clique number that is a function of $m$ and independent of $n$, while the chromatic number is at least $\lceil log_2(n+1) \rceil$. Therefore the ratio $\frac{\chi(\mathcal{G}([n], \mathscr{L}_{m,n}))}{\omega(\mathcal{G}([n], \mathscr{L}_{m,n}))}$ can be bigger than any fixed constant $c$, provided $n$ is large enough.

The precise determination of $\chi(\mathcal{G}([n], \mathscr{L}_{m,n}))$ for $m \geq 2$ is an interesting question in its own right. Let $S$ be a finite set. An $m$-nondecreasing sequence of subsets of $S$ is a sequence $S_1, S_2, \ldots, S_t$ so that for any set $B \subseteq [t]$, $|B| \leq m$, and for $j \in [t]$ with $j > b$ for all $b \in B$, we have $S_j \not\subseteq \bigcup_{b \in B} S_b$.

**Lemma 2.7.** *The chromatic number of $\mathcal{G}([n], \mathscr{L}_{m,n})$ is the smallest integer $s$ for which there is an $m$-nondecreasing sequence of length $n$ of subsets of an $s$-element set $S$.*

*Proof.* For any proper coloring of $\mathcal{G}([n], \mathscr{L}_{m,n})$ with $s$ colors, let $S_i$ be the set of colors used to color the copoints of $A_i$, $i = 1, 2, \ldots, n$. It follows from Corollary 2.5 and the definition of $m$-nondecreasing sequence of subsets of $[s]$, that $S_1, S_2, \ldots, S_n$ is an $m$-sequence of length $n$. Then, it is possible to color the vertices in levels $n, n-1, \ldots, 1$ successively where for any $D = [i-1] \cup J$ in $A_i$ that is adjacent to the copoints in $A_j$ for $j \in J$, there is a color in $S_i$ that does not appear in the label of $S_j$ for $j \in J$. This color can be used for the copoint $D$. Therefore, there is a proper coloring with $s$ colors of $\mathcal{G}([n], \mathscr{L}_{m,n})$. $\square$

We are confronted with the problem of determining the smallest integer $s$ for which there is an $m$-nondecreasing sequence of length $n$ of subsets of an $s$-element set $S$.

A binary covering array of strength $m+1$ is an $s \times n$ matrix $A$ with entries in $\{0, 1\}$ so that for every $s \times m+1$ submatrix $B$ of $A$, every possible $0-1$ vector of length $m+1$ appears as a row of $B$.

**Lemma 2.8.** *If $A$ is an $s \times n$ binary covering array of strength $m+1$, then the columns of $A$ are the characteristic vectors of an $m$-nondecreasing sequence of length $n$ of subsets of an $s$-element set.*

*Proof.* If $A$ is an $s \times n$ binary covering array of strength $m+1$ and $B$ is an $s \times m+1$ submatrix of $A$, then there is a row of $B$ which consists of $m$ zeroes followed by a 1. This implies that the set with characteristic vector equal to column $m+1$ of $B$ is not contained in the union of the sets whose characteristic vectors are the first $m$ columns of $B$. $\square$

The survey paper of Lawrence et. al. [11] on covering arrays gives the result of Kleitman and Spencer [10] that there exist $c_{m+1}$ and $d_{m+1}$ such that the smallest integer $s$ for which there exists an $s \times n$ binary covering array of strength $m+1$ is bounded below by $(c_{m+1} - o(1)) \log n$ and above by $(d_{m+1} + o(1)) \log n$.

**Corollary 2.9.** *There exists a constant $d_{m+1}$ so that the chromatic number of $\mathcal{G}([n], \mathscr{L}_{m,n})$ is at most $(d_{m+1} + o(1)) \log n$.*

2.1. **Order Dimension.** Let $P = (X, \leq)$ be a partially ordered set. The order dimension of $P$, denoted $\dim(P)$, is the least positive integer $t$ for which there exists a family $\mathcal{R} = \{L_1, L_2, \ldots, L_t\}$ of linear extensions of $P$ so that $P = \cap\mathcal{R}$. Any family of linear extensions $\mathcal{R}$ such that $\cap\mathcal{R} = P$ is called a *realizer* of $P$.

Let $\vec{\mathcal{G}}(X, \mathscr{L})$ be a directed graph with vertex set equal to the set of copoints of $(X, \mathscr{L})$ and there is a directed edge from $(\alpha(A), A)$ to $(\alpha(B), B)$ if $\alpha(B) \in A$. From this directed graph, we form a hypergraph $\mathcal{H}(X, \mathscr{L})$ on the same vertex set with $\{(\alpha(A_1), A_1), (\alpha(A_2), A_2), \ldots (\alpha(A_k), A_k)\}$ a hyperedge of $\mathcal{H}(X, \mathscr{L})$ if $\{(\alpha(A_1), A_1), (\alpha(A_2), A_2), \ldots (\alpha(A_k), A_k)\}$ is a minimal directed cycle in $\vec{\mathcal{G}}(X, \mathscr{L})$. Beagley [2] showed that the digraph $\vec{\mathcal{G}}(X, \mathscr{L})$ is isomorphic to one studied by Felsner and Trotter [7],[16] with the possible addition of vertices that do not appear in any directed cycles. It follows from the theory of order dimension that $\dim(L_{\mathscr{L}}) = \chi(\mathcal{H}(X, \mathscr{L}))$. The graph induced by the hyperedges of size 2 in $\mathcal{H}(X, \mathscr{L})$ is $\mathcal{G}(X, \mathscr{L})$.

**Proposition 2.10.** $\dim(L_{\mathscr{L}_{m,n}}) = \chi(\mathcal{G}([n], \mathscr{L}_{m,n}))$

*Proof.* We show that $\mathcal{H}([n], \mathscr{L}_{m,n}) \cong \mathcal{G}([n], \mathscr{L}_{m,n})$. Suppose that there is a hyperedge of size strictly more than 2, $\{(\alpha(B_1), B_1), (\alpha(B_2), B_2), \ldots (\alpha(B_k), B_k)\}$ where $k > 2$. There is some $i \in [k]$ such that $\alpha(B_i) < \alpha(B_{i+1})$ in $[n]$, so $\alpha(B_i) \in B_{i+1}$. Also by definition of $\mathcal{H}([n], \mathscr{L}_{m,n})$, we have that $\alpha(B_{i+1}) \in B_i$. Thus, there is an edge in $\mathcal{G}([n], \mathscr{L}_{m,n})$ between $B_{i+1}$ and $B_i$. So $\{(\alpha(B_1), B_1), (\alpha(B_2), B_2), \ldots (\alpha(B_k), B_k)\}$ was not a hyperedge with $k > 2$. $\square$

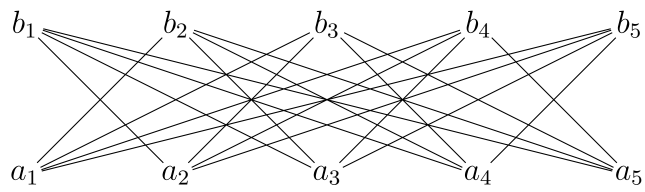


FIGURE 2. The Standard Example, $S_5$

Let $f : (V(\mathcal{G}([n], \mathscr{L}_{1,n}))\backslash\{1, 2, \ldots, n-1\}) \to \binom{[n]}{2}$, where $f([i-1] \cup \{j\}) = \{i, j\}$. Then $f$ is a graph isomorphism from the subgraph of $\mathcal{G}([n], \mathscr{L}_{1,n})$ induced by the vertices other than $\{1, 2, \ldots, n-1\}$ to the shift graph of $K_n$ (see [16],Chapter 8). The shift graph of $K_n$ is known to have clique number 2 and chromatic number $\lceil log_2(n)\rceil$.

In this construction, we have shown that the ratio between the chromatic number and the clique number of the graph $\mathcal{G}([n], \mathscr{L}_{m,n})$ can get arbitrarily large. There is a related result about posets in a book of Trotter ([16]). The *standard example*, $S_n$ for $n \geq 3$, is a partial order on $X = \{a_1, a_2, \ldots, a_n\} \cup \{b_1, b_2, \ldots, b_n\}$ with the relations $a_i < b_j$ if and only if $i \neq j$, for $i, j = 1, 2, \ldots, n$. For $i = 1, 2, \ldots, n$, $a_i$ is a minimal element and $b_i$ is a maximal element of the partial order. Figure 2 is the Hasse diagram of the standard example $S_5$. It is known that the order dimension of $S_n$ is $n$. However, posets with large order dimension do not require $S_n$ as a subposet. Further, [16] gave examples where the ratio between the order dimension of a poset and the largest standard example becomes arbitrarily large. Proposition 2.11 shows that the independent sets of $(X, \mathscr{L})$ in $L_{\mathscr{L}}$ act in much the same manner as $S_n$ in posets.

**Proposition 2.11.** *Let $(X, \mathscr{L})$ be a convex geometry. $L_{\mathscr{L}}$ contains a subposet isomorphic to $S_k$, the standard example, if and only if $\mathcal{G}(X, \mathscr{L})$ contains a $k$-clique.*

*Proof.* We label the standard example $S_k$ contained in $L_{\mathscr{L}}$ in the usual way. Let $p_i$ be a point in $X$ such that $p_i \in (a_i - b_i)$ for $i = 1, 2, \ldots, k$. As $p_i \in a_i$, this means that for all $j \neq i$, $p_i \in b_j$. We now construct the copoint $C_i$ to be a maximal subset of $X - p_i$ containing $b_i$. Consider the copoints $C_i$ and $C_j$ for $j \neq i$. $C_j$ is a copoint attached to $p_j$ and $C_i$ is a copoint attached to $p_i$. By definition, $p_i \in b_j \subseteq C_j$ and $p_j \in b_i \subseteq C_i$. This means that $C_i$ and $C_j$ are adjacent in the graph $\mathcal{G}(X, \mathscr{L})$. Since $C_i$ and $C_j$ are adjacent for all $i \neq j$, we have a clique of size $k$ in $\mathcal{G}(X, \mathscr{L})$.

Conversely, let $\mathcal{G}(X, \mathscr{L})$ contain a $k$-clique composed of copoints $C_1, C_2, \ldots, C_k$ attached to $p_1, p_2, \ldots, p_k$ respectively. By definition of $\mathcal{G}(X, \mathscr{L})$, this means that $p_i \in C_j$ when $i \neq j$. Thus, we let $a_i = \{p_i\}$ and $b_i = C_i$ for $i = 1, 2, \ldots, k$ and we have that $L_{\mathscr{L}}$ contains a subposet isomorphic to $S_k$. $\square$

Theorem 2.6 and Proposition 2.11 together show that the convex geometry $([n], \mathscr{L}_{1,n})$ and its lattice of closed sets, $L_{\mathscr{L}_{1,n}}$, is an example of a poset that has order dimension that becomes arbitrarily large but does not contain a poset isomorphic to $S_3$. The lattice $L_{\mathscr{L}_{1,n}}$ is of order dimension $k$ when $|X| = 2^{k-1}$, which means that $|\mathscr{L}_{1,n}| = 2^{2k-3} + 2^{k-1} + 2^{k-2} + 1$. The example given by Trotter (Example 5.3, [16]) requires a poset of size $R_3(k, 4)$ to have the order dimension equal to $k$, where $R_3(k, 4)$ is the Ramsey number on 3-regular hypergraphs. It is known that $R_3(k, 4)$ is at least $2^{ck\log(k)}$ for some constant $c$ [3]. Thus the posets $L_{\mathscr{L}_{1,n}}$ perform the function of making the order dimension high at a greater economy than do the examples of [16].

2.2. **Remarks.** Convex geometries isomorphic to $([5], \mathscr{L}_{1,5})$ are in the references [4] and [5]. The copoint graph for $([5], \mathscr{L}_{1,5})$ contains an induced 5-cycle. The convex geometry $([5], \mathscr{L}_{2,5})$, for which the copoint graph has clique number 3, shows that 5 elements do not force a 4-clique for general convex geometries even when every 2-element subset is closed. Thus one would need more restrictions for combinatorial analogues of Esther Klein's result that 5 point sets in general position in the plane must contain vertex sets of convex 4-gons. The *chromatic number* of $\mathcal{G}([5], \mathscr{L}_{2,5})$, however, is 4. This will be implied by a Theorem that we prove in the next section.

One can compute that for any $m, n$ the total number of copoints of the convex geometry $([n], \mathscr{L}_{m,n})$ is $\sum_{i=1}^{n} |A_i| = \sum_{i=1}^{n-m} \binom{n-i}{m} + \sum_{i=n-m+1}^{n} 1 = \binom{n}{m+1} + m$. For the case $m = \lfloor \frac{n-1}{2} \rfloor$, we get that the total number of copoints is $\binom{n}{\lfloor \frac{n}{2} \rfloor} + \lfloor \frac{n-1}{2} \rfloor$. In the paper [9] it is stated that no examples of convex geometries with total number of copoints greater than the middle binomial coefficient for the number of elements are known.

## 3. Consequences of Freeness

We now introduce a new problem analogous to the Erdős - Szekeres problem: for any integer $k \geq d \geq 2$, determine the smallest positive integer $K_d(k)$ such that for any $d$-free convex geometry with $|X| \geq K_d(k)$ it follows that $\chi(\mathcal{G}(X, \mathscr{L})) \geq k$. There are two questions of interest related to the study of $K_d(k)$:

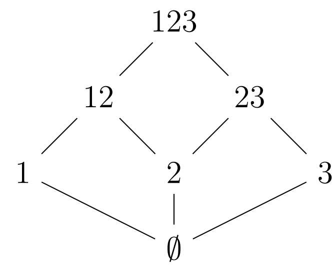


FIGURE 3. Lattice of Closed Sets for a Convex Geometry

**1)** Does the number $K_d(k)$ exist?

**2)** If so, how is $K_d(k)$ determined as a function of $k$?

We specify $d \geq 2$ because of the following 1-free convex geometry. Let $X = [k]$, and for $S \subseteq [k]$ let $\mathscr{L}(S) = [\min(S), \max(S)] \cap X$. Figure 3 shows this convex geometry for $k = 3$. It is clear that there are two chains of copoints for $(X, \mathscr{L})$, those containing 1 and those containing $k$. The graph $\mathcal{G}(X, \mathscr{L})$ has chromatic number 2, for all $k$ for this convex geometry, as each of the chains of copoints is an independent set in $\mathcal{G}(X, \mathscr{L})$. This convex geometry has every 1-element subset closed. Thus 1-freeness alone does not force the chromatic number of the copoint graph to increase with $|X|$.

To show that the number $K_d(k)$ exists for $d > 1$, we focus on the question $K_2(k)$. It is sufficient to show $K_2(k)$ is finite, because $d$-freeness for $d > 2$ implies that every 2-element subset is also closed, because $\mathscr{L}$ is an alignment.

Let $(X, \mathscr{L})$ be a 2-free convex geometry and $\mathcal{I} = \{I_1, I_2, \ldots, I_t\}$ be a partition of $\mathcal{G}(X, \mathscr{L})$ into independent sets. For $x, y \in X, x \neq y$, define $S_{xy} = \{j \in [t] :$ there is a copoint C with $\alpha(C) = y, x \in C, C \in I_j\}$. For each $x \in X$, let $D_x = \{S_{yx} : y \neq x\}$.

A family of subsets of $[t]$ is called *intersecting* if $A \cap B \neq \emptyset$ whenever $A, B \in [t]$. An intersecting family of subsets is *maximal* if it is contained in no other intersecting family.

**Lemma 3.1.** *For each $x \in X$, $D_x$ is an intersecting family.*

*Proof.* A copoint $C$ attached to $x$ is a maximal closed subset in $X - x$. For any $\{y, z\}$ with $x, y$, and $z$ distinct $\{y, z\}$ is closed, so there is a copoint containing $\{y, z\}$ attached to $x$. This copoint must be in one of the independent sets $I_j$. Therefore, $j \in S_{yx} \cap S_{zx}$ and $S_{yx} \cap S_{zx} \neq \emptyset$. $\square$

**Corollary 3.2.** *No two families $D_x$ for $x \in X$ are contained in the same maximal intersecting family of $[t]$.*

*Proof.* For $x \neq y$, $S_{yx}$ is contained in the complement of $S_{xy}$ in $[t]$, because $\mathcal{I}$ is a proper coloring of $\mathcal{G}(X, \mathscr{L})$. □

Results similar to Lemma 3.1 and Corollary 3.2 also appear in [8] and [12]. Moreover, in [12], Morris noted that the number $\gamma(n)$ of maximal intersecting families of subsets of an $n$-element set is at least $2^{\binom{n-1}{\lfloor (n-1)/2 \rfloor}}$, which is a result of Spencer [15].

**Theorem 3.3.** $K_2(k) = \gamma(k)$

*Proof.* Corollary 3.2 shows that $K_2(k) \leq \gamma(k)$. The construction of Hoşten and Morris [8] gives a 2-free convex geometry of convex dimension $k$ with $\gamma(k)$ elements, for any $k$. Edelman and Jamison ([4]) proved that the convex dimension is bounded below by the order dimension and Beagley [2] proved that the order dimension is bounded below by $\chi(\mathcal{G}(X, \mathscr{L}))$. So, $K_2(k) \geq \gamma(k)$. Therefore, $K_2(k) = \gamma(k)$. □

**Corollary 3.4.** *$K_d(k)$ exists for $d \geq 2$, and $K_d(k) \leq 2^{\binom{k-1}{\lfloor (k-1)/2 \rfloor}}$*

The computation of the numbers $K_d(k)$ for $d > 2$ appears to be difficult, in general. We will calculate $K_d(d+2)$. Before we do this, we recall a result of Morris and Soltan [14] to indicate the kind of combinatorial restrictions that lead to analogous results for the clique number. The *Carathéodory number* of a convex geometry $(X, \mathscr{L})$ is the least positive integer $c$ such that $\mathscr{L}(Y) = \cup\{\mathscr{L}(Z) : Z \subseteq Y, |Z| \leq c\}$ for any $Y \subseteq X$.

Let $c$ be the Carathéodory number of a convex geometry $(X, \mathscr{L})$, and suppose that every $c-1$-element subset of $X$ is closed. We say that $(X, \mathscr{L})$ satisfies the simplex partition property if for any set $\{z_1, z_2, \ldots, z_{c+2}\}$ of $c+2$ elements of $X$, the point $z_{c+2}$ belongs to exactly one of the sets $\mathscr{L}(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_c, z_{c+1})$, $i = 1, \ldots, c$. We state a result of Morris and Soltan [14].

**Proposition 3.5** ([14], 5.6). *Let $(X, \mathscr{L})$ be a $(c-1)$-free convex geometry. If $(X, \mathscr{L})$ has Carathéodory number $c$, the simplex partition property, and $|X| = c+2$, then $X$ contains $c+1$ convexly independent points.*

The analogous result for chromatic number does not require the simplex partition property or any condition on the Carathéodory number, only that every $c-1$ element subset be closed.

**Theorem 3.6.** *$K_{c-1}(c+1) = c+2$ for $c \geq 2$.*

*Proof.* The example from Section 2, $([c+1], \mathscr{L}_{c-1,c+1})$, is realizable by a $(c-1)$-simplex with a point in the interior. The copoints of the form $[c+1]\backslash\{i\}$ for $i = 2, 3, \ldots, c+1$ form a $c$-clique. The remaining copoints are $[c+1]\backslash\{1, i\}$ for $i = 2, 3, \ldots, c+1$. For $i = 2, 3, \ldots, c+1$, the copoint $[c+1]\backslash\{1, i\}$ can be colored with the same color as $[c+1]\backslash\{i\}$, so $\chi(\mathcal{G}([c+1], \mathscr{L}_{c-1,c+1})) = c$. Thus $K_{c-1}(c+1) \geq c+2$.

Let $X = \{q_1, q_2, p_1, \ldots, p_c\}$ and assume that $q_1, q_2 \in \mathscr{L}(p_1, \ldots, p_c)$. Further, consider the copoints of $(X, \mathscr{L})$. If there is a convexly independent set of size $c+1$, we have the conclusion. So we may assume that there is no convexly independent set of size $c+1$. There are copoints $(p_i, q_1 q_2 p_1 \ldots p_{i-1} p_{i+1} \ldots p_c)$ for $i = 1, \ldots, c$. Also, there are copoints of the form $(q_{j_i}, q_{k_i} p_1 \ldots p_{i-1} p_{i+1} \ldots p_c)$, because the set $q_1 q_2 p_1 \ldots p_{i-1} p_{i+1} \ldots p_c$ is closed and the set $q_{k_i} p_1 \ldots p_i \ldots p_c$ is not closed. We see that for $i_1, i_2 \in \{1, \ldots c\}, i_1 \neq i_2$, $p_{i_1} \in q_{k_{i_1}} p_1 \ldots p_{i_2-1} p_{i_2+1} \ldots p_c$ and $q_{j_{i_2}} \in q_1 q_2 p_1 \ldots p_{i-1} p_{i+1} \ldots p_c$, so these two copoints are adjacent. Suppose that $\chi(\mathcal{G}(X, \mathscr{L})) = c$, then the following copoints must be colored with the same color: $(p_i, q_1 q_2 p_1 \ldots p_{i-1} p_{i+1} \ldots p_c), (q_{j_i}, q_{k_i} p_1 \ldots p_{i-1} p_{i+1} \ldots p_c)$ for $i = 1, \ldots, c$. Consider the closed set $q_{j_1} p_2 \ldots p_c$, which is a copoint attached to $q_{k_1}$ because $q_{j_1} p_1 p_2 \ldots p_c$ is not a closed set. The copoint $(q_{k_1}, q_{j_1} p_2 \ldots p_c)$ is adjacent to some copoint in each color class because $q_{k_1} \in q_1 q_2 p_1 \ldots p_{i-1} p_{i+1} \ldots p_c$ and $p_i \in q_{j_1} p_2 \ldots p_c$ for $i = 2, \ldots c$. In addition, $q_{k_1} \in q_{k_1} p_2 \ldots p_c$ and $q_{j_1} \in q_{j_1} p_2 \ldots p_c$. Thus, $\chi(\mathcal{G}(X, \mathscr{L})) \geq c+1$. $\square$

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