An Esther Klein Type Coloring Theorem

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Point Sets in General Position

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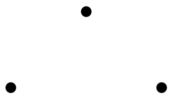
- Observation: Any three points in general position form the vertex set of a triangle



Sets of Five Points in General Position

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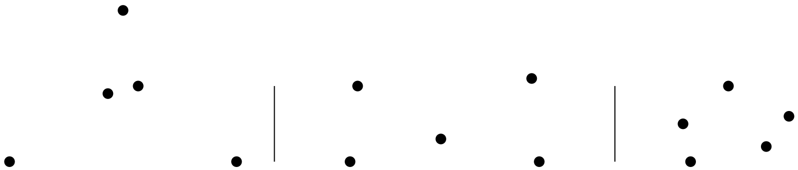

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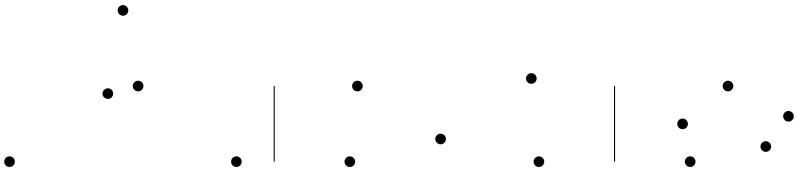


These all contain the vertex set of a convex 4-gon.

Generalized Problem

Problem (Erdős-Szekeres (1935))

For any $n \geq 3$, to determine the smallest positive integer $N(n)$ such that any set of at least $N(n)$ points in general position in the plane (no three points are on a line) contains n points that are the vertices of a convex n-gon.

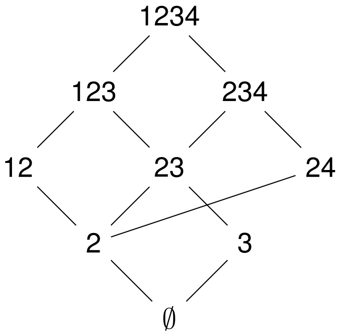
Erdős and Szekeres proved that $N(n)$ is finite (using Ramsey Theory) and in 1961 provided a construction of 2^{n-2} points in general position without the vertex set of a convex n-gon.
It is known that $2^{n-2} + 1 \leq N(n) \leq \binom{2n-5}{n-2} + 1$.

Convex Geometries

- Let X be a finite set, $\mathscr{L}$ a collection of subsets of X with $\emptyset \in \mathscr{L}$, $X \in \mathscr{L}$ and $A \cap B \in \mathscr{L}$ whenever $A, B \in \mathscr{L}$
- $L_{\mathscr{L}} = (\mathscr{L}, \subseteq)$ is a lattice partially ordered by inclusion
- For $C \subseteq X$, define $\mathscr{L}(C)$ to be the intersection of all $A \in \mathscr{L}$ such that $C \subseteq A$
- If $\mathscr{L}(C) = C$, then C is *closed* or *convex*
- For every $C \in \mathscr{L}$, there is a $p \in X \backslash C$ such that $C \cup p \in \mathscr{L}$
- The pair $(X, \mathscr{L})$ where $\mathscr{L}$ has the properties above, is called a *convex geometry*

Examples of Convex Geometries

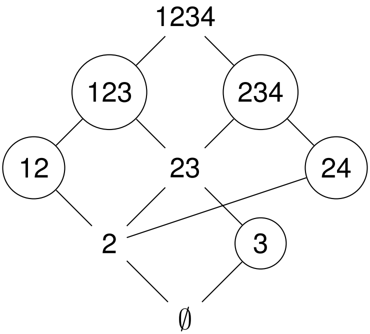
- X a finite set of points in $\mathbb{R}^n$ and for $A \subseteq X$, $\mathscr{L}(A) = \text{conv}(A) \cap X$
- Let T be a graph theoretic tree. $K \subseteq V(T)$ is closed if the subgraph induced by K is connected.



Copoints

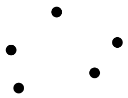
- A closed subset A is a *copoint* if there is exactly one closed subset B such that $|B \backslash A| = 1$
- The set of copoints is $M(X)$
- The unique element in $|B \backslash A|$ is denoted $\alpha(A)$
- We say the copoint A is attached to $\alpha(A)$
- The copoints of a convex geometry realized by point sets in $\mathbb{R}^n$ are subsets of X intersected with open half spaces bounded by a hyperplane through only one point of X

Copoints



Independence

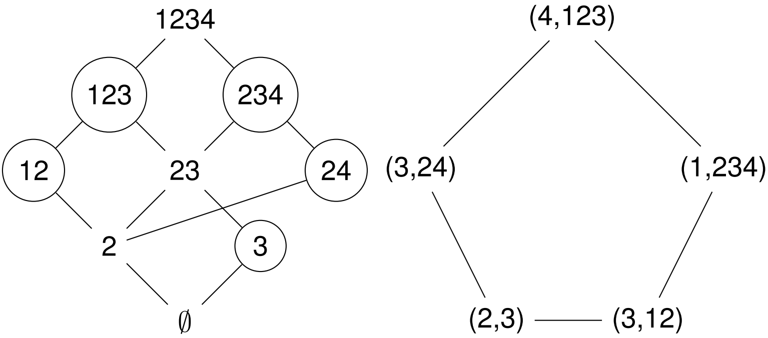
- $B \subseteq X$ is called *(convexly) independent* if for all $p \in B$, $p \notin \mathscr{L}(B \backslash p)$
- For lattice of closed sets $L_{\mathscr{L}} = (\mathscr{L}, \subseteq)$, the size of the largest independent set is $b(L_{\mathscr{L}})$
- The vertex set of a convex n-gon corresponds to an independent set of size n



Graph of Copoints

- Create a graph, $\mathcal{G}(X, \mathscr{L})$, with vertex set equal to $M(X)$ and there is an edge AB if and only if $\alpha(A) \in B$ and $\alpha(B) \in A$
- Morris showed that the cliques in $\mathcal{G}(X, \mathscr{L})$ correspond to independent sets in $(X, \mathscr{L})$
- Beagley showed that the chromatic number of $\mathcal{G}(X, \mathscr{L})$ is related to the order dimension of $L_{\mathscr{L}}$

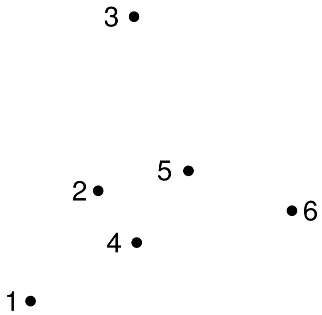
Graph of Copoints



Results for $\mathcal{G}(X, \mathscr{L})$

- Let X is a planar point set in general position and $\mathscr{L}(A) = \text{conv}(A) \cap X$
- Morris showed that if $|X| > 2^{n-2}$, then $\chi(\mathcal{G}(X, \mathscr{L})) \geq n$
- Recall, the ES conjecture is $|X| > 2^{n-2}$, then $\omega(\mathcal{G}(X, \mathscr{L})) \geq n$

Different Clique and Chromatic Numbers



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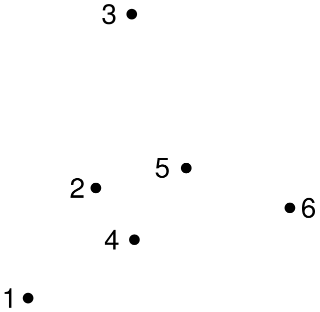

$\chi(\mathcal{G}(X,\mathscr{L})) = 5, \omega(\mathcal{G}(X,\mathscr{L})) = 4$

An Esther Klein Type Theorem

Theorem (B.-Morris)

Let $(X, \mathscr{L})$ be a convex geometry with every two element subset closed. If $|X| = 5$, then $\chi(\mathcal{G}(X, \mathscr{L})) \geq 4$.

This compares with the theorem of Esther Klein that every planar point set in general position of size 5 contains the vertex set of a convex 4-gon.

Generalized EK Type Theorem

Theorem (B.- Morris)

Let $(X, \mathscr{L})$ be a convex geometry and $d \geq 2$ with every d element subset closed. If $|X| = d + 3$, then $\chi(\mathcal{G}(X, \mathscr{L})) \geq d + 2$.

An ES Coloring Problem

Problem

For any integer $n \geq d \geq 2$, determine the smallest positive integer $K_d(n)$ such that any set of $K_d(n)$ points with every d element subset closed requires that $\chi(\mathcal{G}(X, \mathscr{L})) \geq n$.

The last theorem showed that $K_d(d+2) = d+3$. Two important questions can be asked about $K_d(n)$:

1. Does the number $K_d(n)$ exist?
2. If so, how is $K_d(n)$ determined as a function of n?

General Result

- It suffices to show that $K_2(n)$ exists
- Let $\gamma(n)$ be the number of maximal intersecting families of subsets of an n-element set
- A famous result of Spencer states that $\gamma(n) \geq 2^{\binom{n-1}{\lfloor (n-1)/2 \rfloor}}$

Theorem (B.-Morris)

$K_2(n) = \gamma(n)$

The proof is related to the computation of the order dimension of K_n

References

- Beagley, Morris. "Chromatic Numbers of Copoint Graphs of Convex Geometries", *submitted*
  Available at
  http://math.gmu.edu/∼jbeagley/research/CopointGraph.pdf