Properties of the Copoint Graph of Convex Geometries

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March 08, 2013

Outline

1. Introduction
   - Definitions and Motivation
   - Graphs of Copoints
2. Further Developments
   - Minors
   - Different Clique and Chromatic Numbers
   - Direct Sum of Convex Geometries

Motivating Conjecture

Problem (Erdős-Szekeres)

For any $n \geq 3$, to determine the smallest positive integer $N(n)$ such that any set of at least $N(n)$ points in general position in the plane (no three points are on a line) contains n points that are the vertices of a convex n-gon.

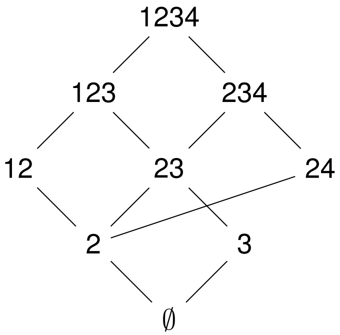
- In 1935, Erdős and Szekeres proved that $N(n)$ is finite and in 1961 provided a construction of 2^{n-2} points in general position without the vertex set of a convex n-gon
- Erdős and Szekeres conjectured that $N(n) = 2^{n-2} + 1$

Convex Geometries

- Let X be a finite set, $\mathscr{L}$ a collection of subsets of X with $\emptyset \in \mathscr{L}$, $X \in \mathscr{L}$ and $A, B \in \mathscr{L}$ implies that $A \cap B \in \mathscr{L}$
- For every $C \in \mathscr{L}$, there is a $p \in X \backslash C$ such that $C \cup p \in \mathscr{L}$
- The pair $(X, \mathscr{L})$ where $\mathscr{L}$ has the properties above, is called a *convex geometry*
- For $C \subseteq X$, define $\mathscr{L}(C)$ to be the intersection of all $A \in \mathscr{L}$ such that $C \subseteq A$
- If $\mathscr{L}(C) = C$, then C is *closed* or *convex*
- $L_{\mathscr{L}} = (\mathscr{L}, \subseteq)$ is a lattice partially ordered by inclusion

Examples of Convex Geometries

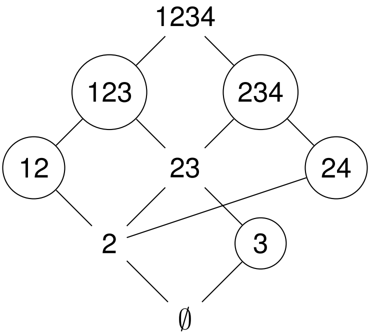
- X a finite set of points in $\mathbb{R}^n$, and for $A \subseteq X$, $\mathscr{L}(A) = \text{conv}(A) \cap X$
- Let T be a tree, $K \subseteq V(T)$ is closed if the induced subgraph on K is connected



Copoints

- A closed subset A is a *copoint* if there is exactly one closed subset B such that $|B \backslash A| = 1$
- The copoints are the meet-irreducible elements of $L_{\mathscr{L}}$
- The set of copoints ordered by inclusion is called $M(X, \mathscr{L})$
- The unique element in $|B \backslash A|$ is denoted $\alpha(A)$
- We say the copoint A is attached to $\alpha(A)$

Copoints



Graph of Copoints

- Create a graph, $\mathcal{G}(X, \mathscr{L})$, with vertex set equal to $M(X, \mathscr{L})$ and there is an edge AB if and only if $\alpha(A) \in B$ and $\alpha(B) \in A$
- Morris (2006) showed that the cliques in $\mathcal{G}(X, \mathscr{L})$ correspond to *convexly independent sets* in $(X, \mathscr{L})$
- B. (to appear in Order) showed that the chromatic number of $\mathcal{G}(X, \mathscr{L})$ is a lower bound for the *order dimension* of $L_{\mathscr{L}}$, $\dim(L_{\mathscr{L}})$

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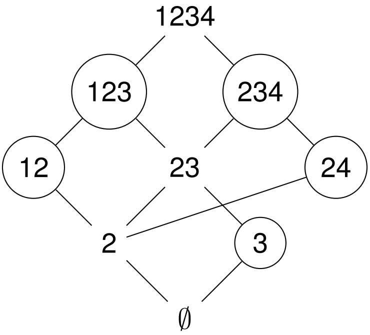
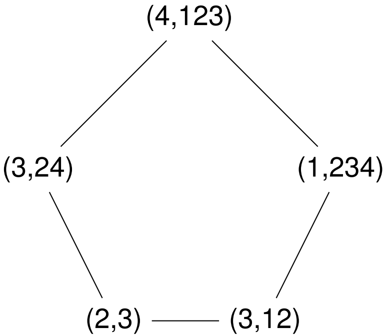
- B. (to appear in Order) showed that the chromatic number of $\mathcal{G}(X, \mathscr{L})$ is a lower bound for the *order dimension* of $L_{\mathscr{L}}$, $\dim(L_{\mathscr{L}})$
- Question: Does $\chi(\mathcal{G}(X, \mathscr{L})) = \dim(L_{\mathscr{L}})$ for all convex geometries $(X, \mathscr{L})$?

Graph of Copoints





Results for Planar Point Sets in General Position

- Let X be a planar point set in general position and $\mathscr{L}(A) = \text{conv}(A) \cap X$
- Morris showed that if $|X| > 2^{n-2}$, then $\chi(\mathcal{G}(X, \mathscr{L})) \geq n$
- Recall, the ES conjecture is $|X| > 2^{n-2}$, implies $\omega(\mathcal{G}(X, \mathscr{L})) \geq n$

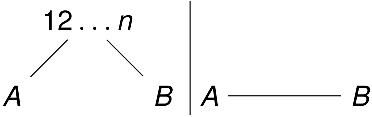
Not every Graph is a Copoint Graph

Theorem (B.)

There is no convex geometry $(X, \mathscr{L})$ such that $\mathcal{G}(X, \mathscr{L})$ is equal to a cycle on 6 or more vertices

Let $|X| = n$, then there must be two copoints of size $n - 1$ otherwise $\mathcal{G}(X, \mathscr{L})$ is disconnected. Call these two copoints A and B.

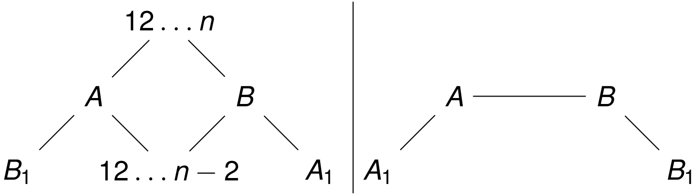
Proof of Theorem



We can say, $\alpha(A) = n$ and $\alpha(B) = n - 1$

Proof of Theorem

There are two more copoints, A_1 and B_1 with adjacent to A and B respectively



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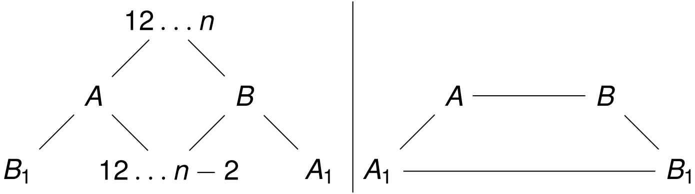

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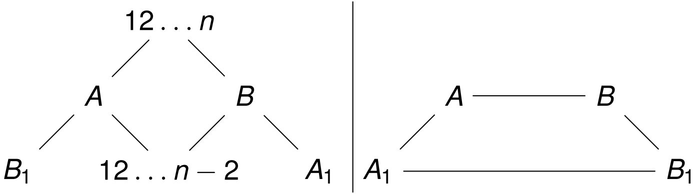
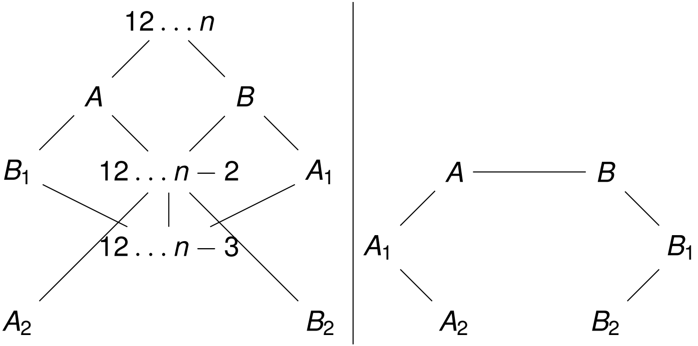


So, $\alpha(A_1) = \alpha(B_1) = n - 2$.

Proof of Theorem

There are two more copoints, A_2, B_2 that are subsets of $\{12 \ldots n-2\}$.



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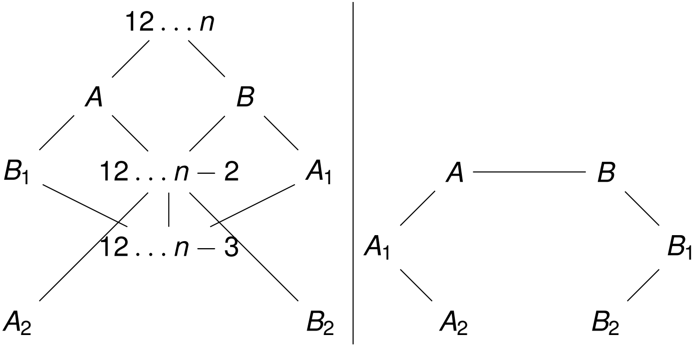
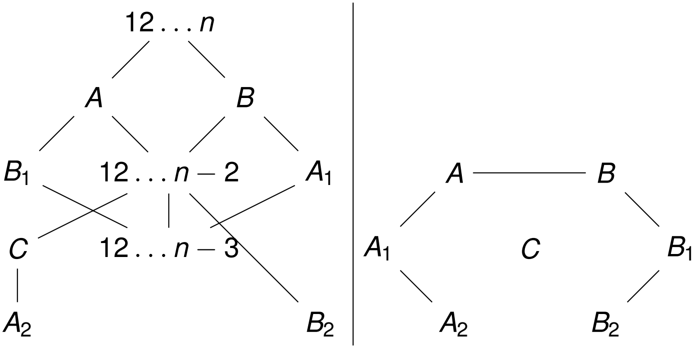


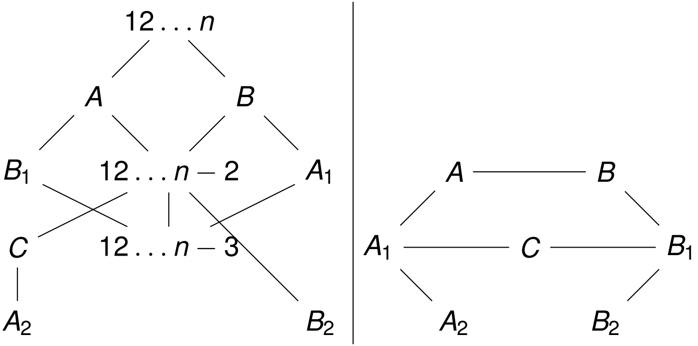
Proof of Theorem

There is a copoint of size $n-3$ containing A_2, call this copoint C.



Proof of Theorem

C does not contain n, $n-1$, and $\alpha(C)$. This means that $\alpha(C) \in A_1$ and $\alpha(C) \in B_1$. But $n-2 \in A_2 \subseteq C$, so:



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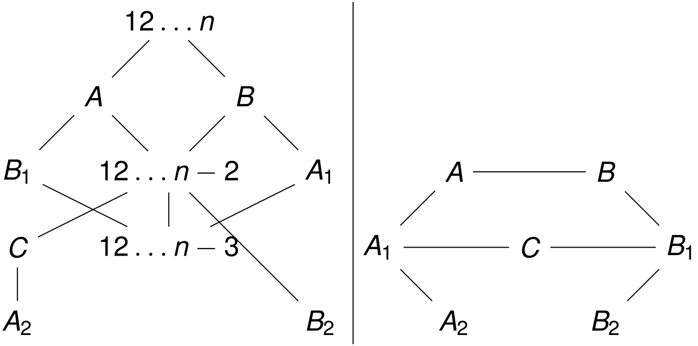


This contains a cycle of size 5. So, there is no convex geometry with $\mathcal{G}(X, \mathscr{L})$ equal to a cycle of size 6 or more.

Restriction

- Let $(X, \mathscr{L})$ be a convex geometry and $Y \subseteq X$
- The restriction of $\mathscr{L}$ to Y, is the alignment $\mathscr{L}|_Y = \{C \cap Y | C \in \mathscr{L}\}$
- The pair $(Y, \mathscr{L}|_Y)$ is a convex geometry.

Restriction and the Copoint Graph

Theorem (B.)

Let $(X, \mathscr{L})$ be a convex geometry with copoint graph $\mathcal{G}(X, \mathscr{L})$. For all $p \in X$,
$\chi(\mathcal{G}(X - p, \mathscr{L}|_{X-p})) + 1 \geq \chi(\mathcal{G}(X, \mathscr{L})) \geq \chi(\mathcal{G}(X - p, \mathscr{L}|_{X-p}))$.

To prove this, we construct graph homomorphisms between $\mathcal{G}(X - p, \mathscr{L}|_{X-p})$ and $\mathcal{G}(X, \mathscr{L})|_{X-p}$.
Also note that $A_p = \{C \in \mathscr{L} : C$ is a copoint attached to $p\}$ is an independent set in $\mathcal{G}(X, \mathscr{L})$.

Contraction

- Let $(X, \mathscr{L})$ be a convex geometry and $Y \subseteq X$ with $Y \in \mathscr{L}$
- The *contraction of* $\mathscr{L}$ *with respect to* Y, is the alignment $\mathscr{L}/Y = \{C \subseteq X - Y | C = \mathscr{L}(D \cup Y) - Y \text{ for some } D \subseteq X - Y\}$
- The pair $(X - Y, \mathscr{L}/Y)$ is a convex geometry.

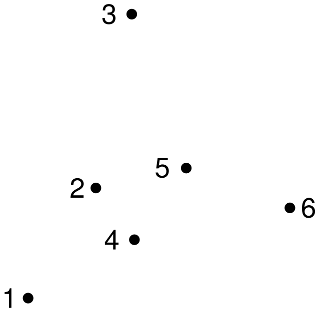
Contraction and the Copoint Graph

Theorem (B.)

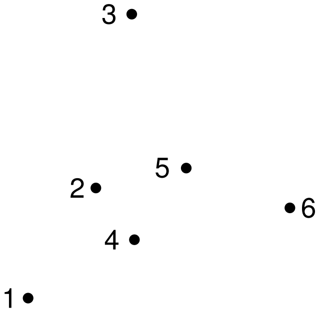
Let $(X, \mathscr{L})$ be a convex geometry with $p \in X$, $p \in \mathscr{L}$. $\mathcal{G}(X-p, \mathscr{L}/p)$ is isomorphic to the subgraph of $\mathcal{G}(X, \mathscr{L})$ induced by the set copoints containing p. Also, $\chi(\mathcal{G}(X, \mathscr{L})) \geq \chi(\mathcal{G}(X-p, \mathscr{L}/p))$.

To prove this, we construct a graph isomorphism between $\mathcal{G}(X-p, \mathscr{L}/p)$ and the induced subgraph of $\mathcal{G}(X, \mathscr{L})$ on the copoints that contain p.

Different Clique and Chromatic Numbers



Different Clique and Chromatic Numbers



$\chi(\mathcal{G}(X,\mathscr{L})) = 5, \omega(\mathcal{G}(X,\mathscr{L})) = 4$

Different Clique and Chromatic Numbers

We computed the chromatic number and clique number for every point set of size 9 or less points in general position from Oswin Aichholzer's Database (http://www.ist.tugraz.at/staff/aichholzer/research/rp/triangulations/ordertypes/)

n	# of Order Types	With Distinct Chromatic and Clique Number
3	1	0
4	2	0
5	3	0
6	16	1
7	135	8
8	3315	85
9	158817	7949

All point sets have $\chi(\mathcal{G}(X,\mathscr{L})) - \omega(\mathcal{G}(X,\mathscr{L})) \in \{0,1\}$.

Question

- There exist planar point sets X in general position with $\chi(\mathcal{G}(X,\mathscr{L})) > \omega(\mathcal{G}(X,\mathscr{L}))$
- Is it possible to construct, for every positive integer m, a planar point set X such that $\chi(\mathcal{G}(X,\mathscr{L})) - \omega(\mathcal{G}(X,\mathscr{L})) > m$?

Question

- There exist planar point sets *X* in general position with $\chi(\mathcal{G}(X, \mathscr{L})) > \omega(\mathcal{G}(X, \mathscr{L}))$
- Is it possible to construct, for every positive integer *m*, a planar point set *X* such that $\chi(\mathcal{G}(X, \mathscr{L})) - \omega(\mathcal{G}(X, \mathscr{L})) > m$?
- Possible to do for a convex geometry.

Direct Sum of Convex Geometries

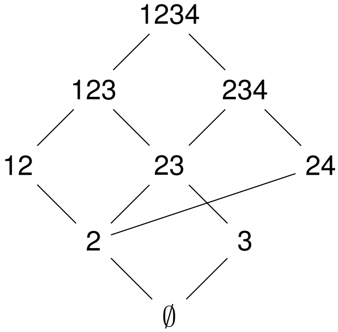
Definition

Let $(X_1, \mathscr{L}_1)$ and $(X_2, \mathscr{L}_2)$ be convex geometries, we define $(X_1, \mathscr{L}_1) \oplus (X_2, \mathscr{L}_2) = (X, \mathscr{L})$ to be the *direct sum* of convex geometries where $X = X_1 \sqcup X_2$ and $\mathscr{L}(C) = \mathscr{L}_1(C_{X_1}) \sqcup \mathscr{L}_2(C_{X_2})$ where $C_{X_i} = C \cap X_i$.

Proposition (B.)

Let $(X_1, \mathscr{L}_1), (X_2, \mathscr{L}_2)$ *be convex geometries and* $(X, \mathscr{L}) = (X_1, \mathscr{L}_1) \oplus (X_2, \mathscr{L}_2)$. *Then,* $\mathcal{G}(X, \mathscr{L}) = \mathcal{G}(X_1, \mathscr{L}_1) \vee \mathcal{G}(X_2, \mathscr{L}_2)$

Direct Sum of Convex Geometries



$\omega(\mathcal{G}(X,\mathscr{L})) = 2, \chi(\mathcal{G}(X,\mathscr{L})) = 3$

Direct Sum of Convex Geometries

Proposition (B.)

For all integers $m \geq 0$, there exists a convex geometry $(X, \mathscr{L})$ such that $\chi(\mathcal{G}(X, \mathscr{L})) - \omega(\mathcal{G}(X, \mathscr{L})) > m$.

- We take the direct sum of the convex geometry on the previous slide *m* times.
- $\chi(\mathcal{G}([n], \mathscr{L})) - \omega(\mathcal{G}([n], \mathscr{L})) = \frac{n}{4}$
- This compares to a construction of B.-Morris where $\frac{\chi(\mathcal{G}([n],\mathscr{L}))}{\omega(\mathcal{G}([n],\mathscr{L}))} = \frac{\lceil log_2(n+1) \rceil}{2}$

Questions?