

Project — Math 413 — Fall 2008

This project must be your own work. No collaboration is allowed. It MUST be in my possession by the start of the final exam. No late projects will be accepted at all since grades have to be submitted within two days of the final exam so there is absolutely no flexibility in this deadline. I prefer to have a hardcopy but will accept electronic submissions if they are in the form of a single pdf file.

Finite Difference Methods are hard to program in the general case, I have tried to keep things simple by making sure that you don't have to worry about changing the finite-difference formulae near boundaries.

Laplace's Equation on a square

We are going to calculate the electrostatic potential $u(x, y)$ in a charge-free square domain $\Omega = (0, 1) \times (0, 1)$. The potential $u(x, y)$ satisfies Laplace's Equation

$$\nabla^2 u(x, y) = 0 \quad \forall (x, y) \in \Omega.$$

The sides $x = 0$, $x = 1$ and $y = 1$ are held at zero potential and the side $y = 0$ has a potential distribution given by $u(x, 0) = x(1 - x)$.

(i) Write a program to numerically approximate the solution to this problem for a variety of different grid spacings h (choosing h to be powers of $\frac{1}{2}$). You will need to encode the process that generates the vector representing u at the gridpoints and the matrix representing the Laplacian operator. You should use MATLAB's inbuilt linear system solvers to solve the resulting linear equations. Produce plots of your solutions.

(ii) By plotting the error at some suitable interior gridpoint for different values of h , formulate a hypothesis as to the rate at which the error tends to zero as $h \rightarrow 0$. (Note: the exact solution to this problem is given by

$$u(x, y) = -4 \sum_{n=1}^{\infty} \left(\frac{\sin(n\pi x)((-1)^n - 1) \sinh(n\pi(1 - y))}{\sinh(n\pi)n^3\pi^3} \right)$$

so use sufficient terms to generate the exact solution to within machine precision and use this as your exact answer.)