# THE KURZWEIL INTEGRAL IN FINANCIAL MARKET MODELING

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Received February 28, 2016
Communicated by Dagmar Medková

*Dedicated to Jaroslav Kurzweil on his ninetieth birthday*

*Abstract.* Certain financial market strategies are known to exhibit a hysteretic structure similar to the memory observed in plasticity, ferromagnetism, or magnetostriction. The main difference is that in financial markets, the spontaneous occurrence of discontinuities in the time evolution has to be taken into account. We show that one particular market model considered here admits a representation in terms of Prandtl-Ishlinskii hysteresis operators, which are extended in order to include possible discontinuities both in time and in memory. The main analytical tool is the Kurzweil integral formalism, and the main result proves the well-posedness of the process in the space of right-continuous regulated functions.

*Keywords*: hysteresis; Prandtl-Ishlinskii operator; Kurzweil integral; market model

*MSC 2010*: 34C55, 26A39, 91B26

## 1. Introduction

Hysteresis and memory in economics has been a subject of interest for some time, see, e.g., [5], [4], [7]. In the recent papers [13], [12] it was shown that the Prandtl-Ishlinskii hysteresis model (which is popular, for example, in the control engineering community for its simplicity and easy numerical implementation for real time control of smart material sensors and actuators, see [15], [1]) can serve as a useful tool for modeling economic processes. In particular, it illustrates a certain analogy between hysteretic memory in the mechanics of plastic materials and in economics. We note

The support by the GAČR Grant GA15-12227S and RVO: 67985840 (PK), by the Institutional Research Plan No. AV0Z10190503 (GAM), and by NSF grant DMS-1413223 (DR) is gratefully acknowledged.

that modeling economic processes using mechanical analogies has a long history (see, for example, [6] for some review of applications of Phillips machines based on fluidic logic).

The Prandtl-Ishlinskii model was introduced in [8], [17] as a linear combination of single-yield mechanical elastoplastic elements called *stops* with unit elasticity modulus and one threshold (yield point), see [9]. Here, we use an equivalent representation based on the dual concept of *play operators*, which are complementary to the stops in the sense that the sum of a stop and a play with the same threshold value is the identity mapping. For our purposes, it is convenient to instead define the play operator with possibly discontinuous inputs as the solution operator of an evolution variational inequality in the Kurzweil integral setting, as in [10]. Note that the concept of the Kurzweil integral goes back to [16], for further information see, e.g., [18], [20].

The main feature of the Prandtl-Ishlinskii model is that all hysteretic trajectories can be represented by a single function called the *primary response curve*, possibly shifted and rotated according to the input history. Moreover, a superposition of two Prandtl-Ishlinskii operators is again a Prandtl-Ishlinskii operator with primary response curve obtained by superposition of the original primary response curves. This property, proved for the case of continuous primary response curves in [11], has been extended to the discontinuous case in [12], where the Kurzweil integral was used with respect to both the time and the memory variables.

The aim of this paper is to set up a rigorous mathematical background for studying discontinuous processes with hysteresis. We first show that a discrete-time process, describing a simple trading strategy, can be represented by a Prandtl-Ishlinskii operator in a Kurzweil integral form. As the main result, we prove that this integral defines a well-posed operator in the space of right-continuous regulated functions of time.

The paper is structured as follows. In Section 2, we present the motivating example of trading strategies and state Proposition 2.1 about the relation between trading strategies and play operators. Section 3 is a survey of known results on the play operator in the space of right-continuous regulated functions, which are used for proving Proposition 2.1 in Section 4. The main result on the well-posedness of the Kurzweil integral definition of the Prandtl-Ishlinskii operator in the space of right-continuous regulated functions is stated as Proposition 5.1 and proved in Section 5. Some consequences for the market model are given in Section 6. In Appendix A we prove an elementary approximation formula for right-continuous BV functions.

## 2. Motivating example

Consider the market in one commodity over the time interval $t \in [0, T]$. Let $q(t)$ be the market price of this commodity at time $t$. We assume that the price is defined by the formula

$$\frac{q(t)}{\bar{p}} = \varrho^{\kappa}(t)\Big(\frac{p(t)}{\bar{p}}\Big), \tag{2.1}$$

where $p(t)$ is the *exogenous information stream* received by the traders; $\bar{p}$ is a fixed currency unit; $\varrho(t) > 0$ is a dimensionless quantity characterizing the *market sentiment* at time $t$; and, $\kappa > 0$ is an empirical exponent. The exogenous information stream is typically modeled by a random process such as, for example, geometric Brownian motion and, in reality, is determined by many factors such as changing production costs, transportation costs, political situations, natural catastrophes, etc.

The model (2.1) is motivated by, for example, [5] where the introduction of a market sentiment term, together with its evolution equation, is offered as a potential explanation for rapid and/or large price movements due to coupling and cascading effects between market participants. This is in contrast to the standard models of mathematical finance that assume the price is only driven by the Brownian (memory-free) exogenous new information.

In practice, financial processes are discrete in time. In this section we shall model them with functions of time which are piecewise-constant and right-continuous. Later, in Section 5, we extend the theory to the space of regulated and right-continuous functions. Recall that a function $f\colon\ [0, T] \to \mathbb{R}$ is said to be *regulated* if both the left and the right limits $f(t-), f(t+)$ exist for each $t \in [0, T]$, with the convention $f(0-) = f(0)$, $f(T+) = f(T)$. The set of right-continuous regulated functions is denoted by $G_R[0, T]$, and endowed with seminorms

$$\|f\|_{[t_1,t_2]} = \sup\{|f(t)|\colon\ t_1 \leqslant t \leqslant t_2\}, \tag{2.2}$$

and with norm $\|f\|_{[0,T]}$; it is a Banach space, with right-continuous piecewise-constant functions as a dense subset.

Let $A$ be the set of *traders* who buy or sell the asset. They do not react to price fluctuations continuously and will have differing approaches to risk-taking and market forecasting. The set of strategies used in practice is vast but in [13] it was argued that a subset of such strategies, based upon recent price changes, serves as a useful proxy for those traders attempting to predict and profit from significant changes in market sentiment.

We introduce *threshold parameters* $d, a \in (0, 1)$, and divide the traders into classes $A_{d,a} \subset A$ parameterized by $d, a$ according to the threshold values in their *trading* *strategy*. A trader $\alpha \in A$ belongs to the class $A_{d,a}$ if his trading strategy with respect to the market price evolution $q \in G_R[0,T]$ is the following:

(a) If $\alpha$ buys the asset at time $t_0$ at price $q(t_0)$, he keeps it until the relative decrease with respect to the maximal value for $t > t_0$ is larger or equal to $d$; that is, the selling time is

$$t_1 = \min\left\{t > t_0\colon \ \frac{q(t)}{\sup\{q(\tau)\colon \ t_0 \leqslant \tau \leqslant t\}} \leqslant 1 - d\right\}.$$

(b) If $\alpha$ sells the asset at time $t_1$ at price $q(t_1)$, he decides to buy it back if the relative increase with respect to the minimal value for $t > t_1$ is larger or equal $a$; that is, the buying time is

$$t_2 = \min\left\{t > t_1\colon \ \frac{q(t)}{\inf\{q(\tau)\colon \ t_1 \leqslant \tau \leqslant t\}} \geqslant 1 + a\right\}.$$

We now introduce the logarithmic input variable $v(t) = \log(p(t)/\overline{p})$, the *logarithmic price* $w(t) = \log(q(t)/\overline{p})$, and the *logarithmic market sentiment* $\sigma(t) = \log \varrho(t)$, so that we have

$$w(t) = v(t) + \kappa\sigma(t). \tag{2.3}$$

The strategies of traders from $A_{d,a}$ in terms of the log-price now read:

(a′) If $\alpha$ buys the asset at time $t_0$ for the log-price $w(t_0)$, the next selling time $t_1$ is defined as the minimum of $t > t_0$ such that

$$w(t) - \sup\{w(\tau)\colon \ t_0 \leqslant \tau \leqslant t\} \leqslant \log(1 - d).$$

(b′) If $\alpha$ sells the asset at time $t_1$ for the log-price $w(t_1)$, the next buying time $t_2$ is defined as the minimum of $t > t_1$ such that

$$w(t) - \inf\{w(\tau)\colon \ t_1 \leqslant \tau \leqslant t\} \geqslant \log(1 + a).$$

Here, for simplicity, we assume that $d = d(r)$ and $a = a(r)$ are functions of one parameter $r$ defined by

$$-\log(1 - d(r)) = \log(1 + a(r)) = r; \tag{2.4}$$

the general case will be treated in a subsequent paper. More specifically, we assume the following rules:

(a$''$) If $\alpha$ buys the asset at time $t_0$ for the log-price $w(t_0)$, the next selling time $t_1$ is defined as the minimum of $t > t_0$ such that

$$w(t) - \sup\{w(\tau)\colon\ t_0 \leqslant \tau \leqslant t\} \leqslant -r.$$

(b$''$) If $\alpha$ sells the asset at time $t_1$ for the log-price $w(t_1)$, the next buying time $t_2$ is defined as the minimum of $t > t_1$ such that

$$w(t) - \inf\{w(\tau)\colon\ t_1 \leqslant \tau \leqslant t\} \geqslant r.$$

All traders in $A_r := A_{d(r),a(r)}$ follow the same strategy, hence, they all simultaneously are or are not in possession of the asset. The fact of possession or non-possession of the asset at time $t$ is described by a function

$$S_r(t) = \begin{cases} +1 & \text{if the traders from } A_r \text{ possess the asset,} \\ -1 & \text{if the traders from } A_r \text{ do not possess the asset.} \end{cases} \tag{2.5}$$

We need to avoid the price becoming infinitely large or infinitely small. Hence, we fix some $w_0 > 0$ sufficiently large and assume that

$$w(t) \in [-w_0, w_0] \quad \forall\, t \in [0, T]. \tag{2.6}$$

For simplicity, we further assume that all traders sold their assets at some moment prior to $t = 0$ for the log-price $-w_0$, and let the history start at $t = 0$. That is,

$$w(0-) = -w_0, \quad S_r(0-) = -1 \quad \forall\, r > 0 \tag{2.7}$$

and traders from $A_r$ start buying as soon as the log-price reaches $-w_0 + r$. Other choices of initial conditions are of course possible but the formulas then become more complicated.

We now show that this model can be interpreted in terms of a hysteresis operator well-known in continuum mechanics, more precisely, the *play operator* $\mathfrak{p}_r$ parameterized by $r > 0$. It was introduced in [9], first for continuous piecewise-monotone inputs and then extended to arbitrary continuous functions by a density argument. More specifically, if $u \in C[0, T]$ is a given function which is monotone (nondecreasing or nonincreasing) in an interval $[t_0, t_1]$, and if the output $\xi_r(t_0) \in [u(t_0) - r, u(t_0) + r]$ is known, then we define $\xi_r(t)$ for $t \in [t_0, t_1]$ by the formula

$$\xi_r(t) = \xi_r(t_0) + P_r(u(t) - \xi_r(t_0)), \tag{2.8}$$

where $P_r\colon \mathbb{R} \to \mathbb{R}$ is the *dead-zone function*

$$P_r(x) = \max\{x - r, \min\{0, x + r\}\} \quad \text{for } x \in \mathbb{R}. \tag{2.9}$$

A convenient way to define the initial condition for $\xi_r$ is to define the *memory state space*

$$\Lambda = \{\lambda \in W^{1,\infty}_{\mathrm{loc}}(0,\infty)\colon\ |\lambda'(r)| \leqslant 1 \text{ a.e.}\}, \tag{2.10}$$

and put

$$\xi_r(0) = \lambda(r) + P_r(u(0) - \lambda(r)) \tag{2.11}$$

for $\lambda \in \Lambda$. We then consider the play operator as a mapping which, with a given memory state $\lambda$ and a given input $u$, produces the output $\xi_r$ and write $\xi_r = \mathfrak{p}_r[\lambda, u]$.

This definition was extended to regulated functions in [3]. Here, we proceed differently and use the variational definition of the play, see (3.1) below. We also choose a special initial condition which fits with the initial condition for the trading strategies stated in the previous paragraph, namely

$$\lambda_0(r) = \min\{-2w_0 + r, 0\}, \tag{2.12}$$

and write for simplicity $\mathfrak{p}_r[u]$ instead of $\mathfrak{p}_r[\lambda_0, u]$ whenever the initial memory state is chosen as in (2.12).

We now state the following result, referring to some background material summarized below in Section 3. The proof will be given at the end of Section 3.

**Proposition 2.1.** *Let* $\mathfrak{p}_r$ *be the play operator defined in* (3.2) *with initial memory state* (2.12). *Let* $w$ *be a given right-continuous piecewise-constant function satisfying* (2.6), *and let* $S_r$ *be the function defined in* (2.5) *by the trading strategy* (a″), (b″). *Then, for every* $t \in [0, T]$ *and every* $r \in (0, 2w_0]$, *we have*

$$S_r(t) = -\frac{\partial^-}{\partial r}\mathfrak{p}_r[2w](t), \tag{2.13}$$

*where* $\partial^-/\partial r$ *denotes the left derivative.*

Remark 2.2. According to (2.6), traders from classes $A_r$ for $r > 2w_0$ are never active in the market and the initial state $S_r(t) = -1$ remains for all times $t \in [0, T]$ and all $r > 2w_0$. This is why we restrict ourselves in Proposition 2.1 to the interval $r \in [0, 2w_0]$ of potentially nontrivial processes.

It remains to define an evolution law for the logarithmic market sentiment $\sigma(t)$. We first consider a simplified model in which the evolution of $\sigma(t)$ is driven only by the relative "strength" of the classes $A_r$. More specifically, we assume that there exists a non-negative nondecreasing function $\psi(r)$ characterizing the *relative weight of the opinion* of the traders in $A_r$, and that

$$\sigma(t) = \int_0^{2w_0} S_r(t)\,\mathrm{d}\psi(r). \tag{2.14}$$

In view of Remark 2.2, it makes no sense to suppose that traders from classes $A_r$ for $r > 2w_0$ have any influence on the market sentiment. Thus we integrate only from 0 to $2w_0$ in (2.14).

Formally, we can use Proposition 2.1 and represent $\sigma$ by play operators, namely

$$\sigma(t) = -\int_0^{2w_0} \frac{\partial^-}{\partial r}\mathfrak{p}_r[2w](t)\,\mathrm{d}\psi(r). \tag{2.15}$$

We will see at the beginning of Section 5 that this is a *Prandtl-Ishlinskii operator* $\mathcal{P}_\psi$ of the form (5.1), with the *primary response curve* $\psi$, that is,

$$\sigma(t) = \mathcal{P}_\psi[2w](t). \tag{2.16}$$

Formula (2.3) now has the form

$$2w(t) = 2v(t) + 2\kappa\mathcal{P}_\psi[2w](t). \tag{2.17}$$

This is an equation for the unknown function $w$ under a given evolution of $v$. It was shown in [12] that it has, for every continuous input $v$, a continuous solution

$$w(t) = \frac{1}{2}(I - 2\kappa\mathcal{P}_\psi)^{-1}(2v)(t)$$

*if and only if* the function $x \mapsto x - 2\kappa\psi(x)$ admits a continuous increasing inverse (cf. Corollary 5.6). If this condition is violated, singularities necessarily occur even if the input stream $v$ is regular. An example will be shown in Section 6.

We can also consider $n$ markets for different assets but driven by one exogenous information stream. We assume that the prices in these markets correlate so that the price in one market is affected by the sentiment of other markets. More precisely, we suppose that the log-price in the $i$-th market is defined by

$$w_i(t) = v(t) + \kappa_i \sum_{j=1}^{n} a_{ij}\sigma_j(t) \tag{2.18}$$

with a non-negative interaction matrix $A = (a_{ij})$ and coefficients $\kappa_i > 0$. Here $\sigma_i$ is the logarithmic sentiment of the $i$-th market

$$\sigma_i(t) = \mathcal{P}_{\psi_i}[2w_i], \tag{2.19}$$

with possibly different functions $\psi_i$. The log-prices $w_i(t)$ are then determined as solutions of the system

$$2w_i(t) = 2v(t) + 2\kappa_i \sum_{j=1}^{n} a_{ij}\mathcal{P}_{\psi_j}[2w_j](t), \quad i = 1, \ldots, n, \tag{2.20}$$

which is a vector counterpart of (2.17). The solvability of such systems was discussed in [12].

## 3. Play operator

The main goal of this section is to give some preliminary lemmas for the proof of Proposition 2.1. We first survey known results on the play operator $\mathfrak{p}_r$ with threshold $r > 0$. The parameter $r$ plays the role of *memory variable* and loosely correlates to the memory depth of the system. For a right-continuous regulated input $u \in G_R[0,T]$, an initial memory state $\lambda \in \Lambda$, see (2.10), and a parameter $r > 0$, consider the variational inequality for the unknown function $\xi_r \in BV_R[0,T]$:

$$\begin{cases} \xi_r(0) = \lambda(r) + P_r(u(0) - \lambda(r)), & \\ |u(t) - \xi_r(t)| \leqslant r & \forall\, t \in [0,T], \\ \int_0^T (u(t) - \xi_r(t) - y(t))\,\mathrm{d}\xi_r(t) \geqslant 0 & \forall\, y \in G[0,T],\ |y(t)| \leqslant r,\ \forall\, t \in [0,T], \end{cases} \tag{3.1}$$

where $P_r$ is the dead-zone function (2.9). The solution $\xi_r \in BV_R[0,T]$ of (3.1) exists and is unique, see [14]. This enables us to define the play as the solution mapping

$$\mathfrak{p}_r \colon\ \Lambda \times G_R[0,T] \to BV_R[0,T] \colon\ u \mapsto \xi_r, \tag{3.2}$$

and we write $\xi_r(t) = \mathfrak{p}_r[\lambda, u](t)$. Moreover, the play is Lipschitz continuous with respect to the sup-norm. More specifically, for $\lambda, \mu \in \Lambda$ and $u, v \in G_R[0,T]$ (see [14]):

$$|\mathfrak{p}_r[\lambda, u](t) - \mathfrak{p}_r[\mu, v](t)| \leqslant \max\{|\lambda(r) - \mu(r)|, \|u - v\|_{[0,t]}\}. \tag{3.3}$$

As a consequence, we also have for each $0 \leqslant t < t + h \leqslant T$ that

$$|\mathfrak{p}_r[\lambda, u](t+h) - \mathfrak{p}_r[\lambda, u](t)| \leqslant \|u(\cdot) - u(t)\|_{[t,t+h]}. \tag{3.4}$$

In particular, for a piecewise-constant input

$$u(t) = \sum_{j=1}^{m} u_{j-1}\chi_{[t_{j-1},t_j)}(t) + u_m\chi_{\{t_m\}}(t), \tag{3.5}$$

corresponding to a division $0 = t_0 < t_1 < \ldots < t_m = T$ of the interval $[0,T]$, with given real values $u_0, u_1, \ldots, u_m$, the solution $\xi_r$ of (3.1) has the same form

$$\xi_r(t) = \sum_{j=1}^{m} \xi_{j-1}(r)\chi_{[t_{j-1},t_j)}(t) + \xi_m(r)\chi_{\{t_m\}}(t), \tag{3.6}$$

where $\chi_B$ is the characteristic function of a set $B \subset \mathbb{R}$, that is, $\chi_B(r) = 1$ if $r \in B$, $\chi_B(r) = 0$ if $r \notin B$. The coefficients $\xi_j$ in (3.6) are given by the recursive formula

$$\xi_j(r) = \xi_{j-1}(r) + P_r(u_j - \xi_{j-1}(r)), \tag{3.7}$$

with $\xi_0(r)$ and $P_r$ as in (3.1) and (2.9), respectively.

We now state and prove a few technical lemmas which will be useful in the sequel.

**Lemma 3.1.** *Let $\lambda \in \Lambda$ and $u \in G_R[0,T]$ be given, and let $\xi_r = \mathfrak{p}_r[\lambda, u]$. Let there exist $h > 0$, $t \in [0, T-h]$, and $\omega > 0$ such that for all $\tau \in [t, t+h]$*

$$u(\tau) - \xi_r(\tau) - r \leqslant -\omega. \tag{3.8}$$

*Then, the function $\tau \mapsto \mathfrak{p}_\varrho[\lambda, u](\tau)$ is nonincreasing in $[t, t+h]$ for every $\varrho \geqslant r$. Similarly, if for all $\tau \in [t, t+h]$*

$$u(\tau) - \xi_r(\tau) + r \geqslant \omega, \tag{3.9}$$

*then the function $\tau \mapsto \mathfrak{p}_\varrho[\lambda, u](\tau)$ is nondecreasing in $[t, t+h]$ for every $\varrho \geqslant r$.*

P r o o f. Let (3.8) hold. By [10], Lemma 2.2 we have for all $t \leqslant a < b \leqslant t+h$ that

$$\int_a^b (u(\tau) - \xi_r(\tau) - y(\tau))\,\mathrm{d}\xi_r(\tau) \geqslant 0 \quad \forall\, y \in G[a,b],\ |y(\tau)| \leqslant r,\ \forall\, \tau \in [a,b]. \tag{3.10}$$

In particular, $y(\tau) = u(\tau) - \xi_r(\tau) + \omega$ is an admissible choice. Then (3.10) yields

$$-\omega \int_a^b \mathrm{d}\xi_r(\tau) = \omega(\xi_r(a) - \xi_r(b)) \geqslant 0. \tag{3.11}$$

We thus have proved that $\tau \mapsto \mathfrak{p}_r[\lambda, u](\tau)$ is nonincreasing in $[t, t+h]$. To check that the assertion holds for all $\varrho \geqslant r$, it suffices to realize that the function $\varrho \mapsto \varrho + \mathfrak{p}_\varrho[\lambda, u](t)$ is nondecreasing. Hence, if (3.8) holds for some $r$, then it holds for all $\varrho \geqslant r$. The case (3.9) is similar. $\square$

**Lemma 3.2.** *Let $\lambda, \mu \in \Lambda$ and $u, v \in G_R[0,T]$ be given, and let $\xi_r = \mathfrak{p}_r[\lambda, u]$, $\eta_r = \mathfrak{p}_r[\mu, v]$. Assume that $u(t) \geqslant v(t)$ for all $t$ in an interval $[a,b] \subset [0,T]$, and that $\xi_r(a) \geqslant \eta_r(a)$. Then $\xi_r(t) \geqslant \eta_r(t)$ for all $t \in [a,b]$.*

P r o o f. Every right-continuous regulated function can be uniformly approximated by step functions of the form (3.5) and the play operator is continuous with respect to uniform convergence, see [14]. Hence, it suffices to prove the statement for piecewise-constant functions $u, v$ as in (3.5). More precisely, we prove that if for some $j$ we have

$$\begin{aligned} \xi_j(r) &= \xi_{j-1}(r) + P_r(u_j - \xi_{j-1}(r)), \\ \eta_j(r) &= \eta_{j-1}(r) + P_r(v_j - \eta_{j-1}(r)), \end{aligned}$$

and $u_j \geqslant v_j$, $\xi_{j-1}(r) \geqslant \eta_{j-1}(r)$, then $\xi_j(r) \geqslant \eta_j(r)$. Lemma 3.2 then follows by induction. We have

$$\xi_j(r) - \eta_j(r) = u_j - v_j - (I - P_r)(u_j - \xi_{j-1}(r)) + (I - P_r)(v_j - \eta_{j-1}(r)).$$

The function $(I - P_r)(x) = \max\{-r, \min\{x, r\}\}$ is nondecreasing and Lipschitz continuous with Lipschitz constant 1. In particular, $(I - P_r)(x) - (I - P_r)(y) \geqslant 0$ if $x \geqslant y$, $(I - P_r)(x) - (I - P_r)(y) \geqslant x - y$ if $x \leqslant y$. Hence, we have either

$$u_j - \xi_{j-1}(r) \leqslant v_j - \eta_{j-1}(r) \Longrightarrow \xi_j(r) - \eta_j(r) \geqslant u_j - v_j \geqslant 0,$$

or

$$u_j - \xi_{j-1}(r) \geqslant v_j - \eta_{j-1}(r) \Longrightarrow \xi_j(r) - \eta_j(r) \geqslant \xi_{j-1}(r) - \eta_{j-1}(r) \geqslant 0,$$

and the assertion follows. □

## 4. Proof of Proposition 2.1

The proof of Proposition 2.1 will be carried out in several steps. We fix a right-continuous piecewise-constant function $w\colon [0,T] \to [-w_0, w_0]$, a parameter $r > 0$, and a time $t \in [0,T]$, and find all switching points $0 \leqslant t_1 < \ldots < t_n \leqslant t$ of $S_r(\tau)$ in the interval $[0,t]$, and include an artificial "switching" point $t_0 < 0$ as a starting point. By the choice (2.7) of the initial conditions, $S_r$ switches from $(-1)^j$ to $(-1)^{j+1}$ at the point $t_j$, $j = 0, 1, \ldots, n$. The following two lemmas deal separately with the switching points and with the intermediate points.

**Lemma 4.1.** *For all $j = 0, 1, \ldots, n$ and all $\varrho \in [0, r]$ we have*

$$\mathfrak{p}_\varrho[2w](t_j) = 2w(t_j) + (-1)^j \varrho.$$

P r o o f. The statement is true for $j=0$ by (2.7). We continue by induction and assume that it holds for $j-1$. Let us assume for definiteness that $j$ is odd, the other case is fully analogous.

Put $w^\flat = \inf\limits_{[t_{j-1},t_j]} w$, and let $t_* \in [t_{j-1}, t_j]$ be such that $w^\flat = \min\{w(t_*-), w(t_*)\}$. We define an auxiliary function

$$w^*(t) = \begin{cases} w^\flat & \text{for } t \in [t_{j-1}, t_*), \\ w(t_j) & \text{for } t \in [t_*, t_j], \end{cases} \tag{4.1}$$

and set

$$\begin{aligned} \mathfrak{p}_r[2w^*](t_{j-1}) &= 2w^\flat + r && \text{if } t_* > t_{j-1}, \\ \mathfrak{p}_r[2w^*](t_{j-1}) &= 2w(t_j) - r && \text{if } t_* = t_{j-1}. \end{aligned}$$

For $t_* > t_{j-1}$ we have by the induction hypothesis

$$\mathfrak{p}_r[2w](t_{j-1}) = 2w(t_{j-1}) + r \geqslant \mathfrak{p}_r[2w^*](t_{j-1}),$$

and

$$w(t) \geqslant w^*(t) \text{ in } [t_{j-1}, t_*).$$

Hence, by Lemma 3.2, $\mathfrak{p}_r[2w](t) \geqslant \mathfrak{p}_r[2w^*](t)$ in $[t_{j-1}, t_*)$. In particular,

$$2w^\flat + r = \mathfrak{p}_r[2w^*](t_*-) \leqslant \mathfrak{p}_r[2w](t_*-) \leqslant 2w(t_*-) + r.$$

We further have

$$\mathfrak{p}_r[2w^*](t_*) = \max\{2w(t_j) - r, 2w^\flat + r\} = 2w(t_j) - r$$

from the fact that $t_j$ is a switching point of $S_r$, and so

$$\mathfrak{p}_r[2w](t_*) = \max\{2w(t_*) - r, \min\{\mathfrak{p}_r[2w](t_*-), 2w(t_*) + r\}\}.$$

We have either $w(t_*-) = w^\flat$, $\mathfrak{p}_r[2w](t_*-) = 2w^\flat + r$, or $w(t_*) = w^\flat$ and $\mathfrak{p}_r[2w](t_*-) \geqslant 2w^\flat + r$. In both cases we obtain

$$\mathfrak{p}_r[2w](t_*) = \max\{2w(t_*) - r, 2w^\flat + r\} \leqslant \mathfrak{p}_r[2w^*](t_*).$$

Furthermore, $w(t) \leqslant w^*(t)$ in $[t_*, t_j]$. Hence, by Lemma 3.2,

$$\mathfrak{p}_r[2w](t_j) \leqslant \mathfrak{p}_r[2w^*](t_j) = 2w(t_j) - r.$$

By definition of the play, we always have $\mathfrak{p}_r[2w](t_j) \geqslant 2w(t_j) - r$. Hence, $\mathfrak{p}_r[2w](t_j) = 2w(t_j) - r$. Since $\mathfrak{p}_0[2w](t_j) = 2w(t_j)$ and $\mathfrak{p}_r$ is Lipschitz continuous in $r$ with Lipschitz constant 1, we obtain $\mathfrak{p}_\varrho[2w](t_j) = 2w(t_j) - \varrho$ for all $\varrho \in [0, r]$. □

**Lemma 4.2.** *Assume that the interval* $(t_n, t]$ *does not contain any switching point of* $S_r$*, and assume for definiteness that* $n$ *is even. Then there exists a* $\delta > 0$ *such that* $\mathfrak{p}_\varrho[2w](t) = 2w^\flat + \varrho$ *for all* $\varrho \in [r - \delta, r]$*.*

P r o o f. As in the proof of Lemma 4.1, let $t_* \in [t_n, t]$ be such that $w^\flat := \inf_{[t_n,t]} w = \min\{w(t_*-), w(t_*)\}$. By virtue of the trading strategy (b″), we have $w(\tau) < w^\flat + r$ for all $\tau \in [t_n, t]$. Since $w$ is piecewise constant, there exists $\delta > 0$ such that

$$\tau \in [t_*, t] \ \Rightarrow \ w(\tau) \leqslant w^\flat + r - \delta. \tag{4.2}$$

We use again Lemma 3.2 and define auxiliary functions

$$w^*(\tau) = \begin{cases} w^\flat & \text{for } \tau \in [t_n, t_*), \\ w^\flat + r - \delta & \text{for } \tau \in [t_*, t], \end{cases} \qquad w_*(\tau) = w^\flat \quad \text{for } \tau \in [t_n, t], \tag{4.3}$$

and set $\mathfrak{p}_\varrho[2w^*](t_n) = \mathfrak{p}_\varrho[2w_*](t_n) = 2w^\flat + \varrho$ for $\varrho \in [r - \delta, r]$. If $t_* > t_n$, then we argue as in the proof of Lemma 4.1 and obtain

$$2w^\flat + \varrho = \mathfrak{p}_\varrho[2w^*](t_*-) \leqslant \mathfrak{p}_\varrho[2w](t_*-) \leqslant 2w(t_*-) + \varrho.$$

By virtue of (4.2) we have for $\varrho \in [r - \delta, r]$ in this case that

$$\mathfrak{p}_\varrho[2w^*](t_*) = \max\{2w(t_*) - \varrho, 2w^\flat + \varrho\} = 2w^\flat + \varrho,$$

and similarly

$$\mathfrak{p}_\varrho[2w](t_*) = \max\{2w(t_*) - \varrho, \min\{\mathfrak{p}_\varrho[2w](t_*-), 2w(t_*) + \varrho\}\} = 2w^\flat + \varrho.$$

In $[t^*, t]$ we have by Lemma 3.2 that

$$2w^\flat + \varrho = \mathfrak{p}_\varrho[2w_*](\tau) \leqslant \mathfrak{p}_\varrho[2w](\tau) \leqslant \mathfrak{p}_\varrho[2w^*](\tau) = 2w^\flat + \varrho,$$

which we wanted to prove. □

P r o o f of Proposition 2.1. Let $w$ be a given right-continuous piecewise-constant function and let $t \in [0, T]$ be given. By Lemmas 4.1, 4.2 for each $r \in (0, \infty)$ there exists $\delta(r) \in [0, r)$ such that for all $\varrho \in [r - \delta(r), r]$ we have $S_\varrho(t) = -(\partial/\partial\varrho)\mathfrak{p}_\varrho[2w](t)$. In particular, for every $r \in (0, \infty)$ we have

$$S_r(t) = -\frac{\partial^-}{\partial r}\mathfrak{p}_r[2w](t), \tag{4.4}$$

which completes the proof. □

## 5. Prandtl-Ishlinskii operator

Let $\psi\colon\ [0,\infty) \to \mathbb{R}$ with $\psi(0) = 0$ be an arbitrary right-continuous function with bounded variation. The Prandtl-Ishlinskii operator $\mathcal{P}_\psi\colon\ \Lambda \times G_R[0,T] \to G_R[0,T]$ generated by $\psi$ is defined by the Kurzweil integral formula

$$\mathcal{P}_\psi[u](t) = -\int_0^\infty \frac{\partial^-}{\partial r}\mathfrak{p}_r[u](t)\,\mathrm{d}\psi(r). \tag{5.1}$$

The function $\psi$ is called the *primary response curve* of $\mathcal{P}_\psi$. Note that the play $\mathfrak{p}_{r_0}$ with threshold $r_0$ can be considered as a special case of the Prandtl-Ishlinskii operator with the choice $\psi(r) = (r - r_0)^+$.

To see that the integral in (2.15) coincides with (5.1) for $u = 2w$ under the assumption (2.6), it suffices to note that in (5.1) we have $\mathfrak{p}_r[u](t) = 0$ for all $r \geqslant 2w_0$ and all $t \in [0,T]$. Indeed, in the definition (3.1) of the play, we choose the test function

$$y(t) = \chi_{[0,t_0]}(t)u(t) + \chi_{(t_0,T]}(t)(u(t) - \xi_r(t)) \tag{5.2}$$

with an arbitrarily fixed time $t_0 \in (0,T]$. For $r \geqslant 2w_0$, (5.2) is admissible. Then we have

$$\begin{aligned} 0 &\leqslant \int_0^T (u(t) - \xi_r(t) - y(t))\,\mathrm{d}\xi_r(t) \\ &= -\int_0^{t_0} \xi_r(t)\,\mathrm{d}\xi_r(t) - \int_{t_0}^T \chi_{\{t_0\}}(t)\xi_r(t_0)\,\mathrm{d}\xi_r(t) \\ &= -\int_0^{t_0} \xi_r(t)\,\mathrm{d}\xi_r(t). \end{aligned} \tag{5.3}$$

By the right-continuity of $\xi_r$ we have

$$\int_0^{t_0} \xi_r(t)\,\mathrm{d}\xi_r(t) = \frac{1}{2}(\xi_r^2(t_0) - \xi_r^2(0)) + \frac{1}{2}\sum_{t\in[0,t_0]}(\xi_r(t) - \xi_r(t-))^2. \tag{5.4}$$

We have by hypothesis that $\xi_r(0) = 0$ for $r \geqslant 2w_0$, so that (5.3)–(5.4) imply $\xi_r(t_0) = 0$ for all $t_0 \in (0,T]$ and all $r \geqslant 2w_0$, which we wanted to check.

Formula (5.1) extends the classical definition of the Prandtl-Ishlinskii operator. If $\psi$ is differentiable and its right derivative $\psi'_+$ is regulated, then we can integrate by parts and rewrite (5.1) as

$$\mathcal{P}_\psi[u](t) = \psi'_+(0)u(t) + \int_0^\infty \mathfrak{p}_r[u](t)\,\mathrm{d}\psi'_+(r). \tag{5.5}$$

For example, if $\psi'_+$ is piecewise constant with jumps at points $r_j$, then (5.5) reads

$$\mathcal{P}_\psi[u](t) = a_0 u(t) + \sum_{j=1}^{n} a_j \mathfrak{p}_{r_j}[u](t), \tag{5.6}$$

which corresponds to the original construction in [17] as a finite linear combination of simple elastoplastic elements.

In this section, we prove the following statement which guarantees that formula (5.1) is meaningful.

**Proposition 5.1.** *Let $u \in G_R[0,T]$ be given such that $\|u\|_{[0,T]} < 2w_0$, and let $\psi$ be a nondecreasing right continuous function, $\psi(0) = 0$. Then the function $\sigma$ defined for $t \in [0,T]$ by the Kurzweil integral*

$$\sigma(t) = -\int_0^\infty \frac{\partial^-}{\partial r} \mathfrak{p}_r[u](t)\,\mathrm{d}\psi(r) \tag{5.7}$$

*belongs to $G_R[0,T]$.*

We split the proof of Proposition 5.1 into several steps. We first show that the value of $\sigma(t)$ is well-defined.

**Lemma 5.2.** *Under the hypotheses of Proposition* 5.1*, the integral on the right-hand side of* (5.7) *exists in the Kurzweil sense for every $t \in [0,T]$.*

P r o o f. Let $t \in [0,T]$ be fixed, and put $r_0 = \sup_{\tau \in [0,t]} u(\tau)$. By [3], Proposition 2.7.6 there exists a sequence $\{r_j\}_{j=0}^\infty$, called the memory sequence of $u$ at time $t$, such that $r_0 > r_1 > \ldots > r_n \geqslant r_{n+1} \geqslant \ldots \geqslant 0$, and either $r_{n+1} = 0$ or $r_{j-1} > r_j$ for all $j \in \mathbb{N}$ and $\lim_{j\to\infty} r_j = 0$, with the property that

$$\mathfrak{p}_r[u](t) = \begin{cases} \lambda_0(r) & \text{for } r \geqslant r_0, \\ \lambda_0(r_0) + \sum\limits_{i=0}^{j-1} (-1)^i (r_i - r_{i+1}) + (-1)^j (r_j - r) & \text{for } r \in [r_{j+1}, r_j]. \end{cases} \tag{5.8}$$

In particular, we have

$$u(t) = \lambda_0(r_0) + \sum_{i=0}^{\infty} (-1)^i (r_i - r_{i+1}). \tag{5.9}$$

Note that by the choice of $\lambda_0$ in (2.12) and by the condition $\|u\|_{[0,T]} < 2w_0$, we always have $2w_0 > r_0 > r_1$. From (5.8) it follows that

$$\frac{\partial^-}{\partial r} \mathfrak{p}_r[u](t) = \begin{cases} 0 & \text{for } r > 2w_0, \\ 1 & \text{for } r \in (r_0, 2w_0], \\ (-1)^{j+1} & \text{for } r \in (r_{j+1}, r_j],\ j = 0, 1, \ldots \end{cases} \tag{5.10}$$

The value of $p_0 = (\partial^-/\partial r)\mathfrak{p}_r[u](0)$ can be chosen arbitrarily. Indeed, we have

$$\int_0^\infty p_0 \chi_{\{0\}}(r)\,\mathrm{d}\psi(r) = p_0(\psi(0+) - \psi(0)) = 0.$$

Then (5.7) can be written as

$$\sigma(t) = \int_0^\infty \sum_{j=-1}^{\infty} (-1)^j \chi_{(r_{j+1}, r_j]}(r)\,\mathrm{d}\psi(r) \tag{5.11}$$

with the convention $r_{-1} = 2w_0$, provided we prove that the integral on the right-hand side of (5.11) exists.

Put $f(r) = \sum_{j=-1}^{\infty} (-1)^j \chi_{(r_{j+1}, r_j]}(r)$ and $\psi^{(n)}(r) = \chi_{[r_n, \infty)}(r)\psi(r)$ for $r \geqslant 0$. Note that $f(0+)$ does not exist if the sequence $\{r_j\}$ is infinite, so that $f$ is possibly not regulated. We further have $\psi(r) - \psi^{(n)}(r) = \chi_{[0, r_n)}(r)\psi(r)$, hence

$$\lim_{n\to\infty} \operatorname*{Var}_{[0,\infty)} (\psi - \psi^{(n)}) = 0. \tag{5.12}$$

For every $n \in \mathbb{N}$ we have the explicit formula

$$\begin{aligned} \int_0^\infty f(r)\,\mathrm{d}\psi^{(n)}(r) &= (-1)^n \psi(r_n) + \int_{r_n}^\infty \sum_{j=-1}^{n-1} (-1)^j \chi_{(r_{j+1}, r_j]}(r)\,\mathrm{d}\psi^{(n)}(r) \\ &= (-1)^n \psi(r_n) + \sum_{j=-1}^{n-1} (-1)^j (\psi(r_j) - \psi(r_{j+1})). \end{aligned} \tag{5.13}$$

We thus have

$$\lim_{n\to\infty} \int_0^\infty f(r)\,\mathrm{d}\psi^{(n)}(r) = \sum_{j=-1}^{\infty} (-1)^j (\psi(r_j) - \psi(r_{j+1})), \tag{5.14}$$

which is a convergent series. Together with (5.12), we may use [19], Theorem 4.18 to conclude that the integral in (5.7) exists and equals

$$\sigma(t) = -\int_0^\infty \frac{\partial^-}{\partial r}\mathfrak{p}_r[u](t)\,\mathrm{d}\psi(r) = \sum_{j=-1}^{\infty} (-1)^j (\psi(r_j) - \psi(r_{j+1})). \tag{5.15}$$

$\square$

**Lemma 5.3.** *Under the hypotheses of Proposition* 5.1, *the function* $\sigma\colon [0,T] \to \mathbb{R}$ *defined by* (5.7) *is regulated.*

P r o o f. With the sequence $\{\psi_k\}$ constructed in Proposition A.1 (see Appendix), we define the functions

$$\sigma_k(t) = -\int_0^\infty \frac{\partial^-}{\partial r}\mathfrak{p}_r[u](t)\,\mathrm{d}\psi_k(r) = \int_0^\infty \mathfrak{p}_r[u](t)\,\mathrm{d}\psi_k'(r). \tag{5.16}$$

It follows from (3.4) that the functions $\sigma_k$ are regulated and right-continuous. We now prove that they converge pointwise to $\sigma$. We have by (5.15) that

$$\sigma_k(t) - \sigma(t) = \sum_{j=-1}^{\infty} (-1)^j((\psi_k(r_j) - \psi(r_j)) - (\psi_k(r_{j+1}) - \psi(r_{j+1}))). \tag{5.17}$$

Hence, for each $n \in \mathbb{N}$,

$$\begin{aligned} |\sigma_k(t) - \sigma(t)| \leqslant{} & \sum_{j=n}^{\infty}(|\psi_k(r_j) - \psi_k(r_{j+1})| + |\psi(r_j) - \psi(r_{j+1})|) \\ & + \Bigg|\sum_{j=-1}^{n-1} (-1)^j((\psi_k(r_j) - \psi(r_j)) - (\psi_k(r_{j+1}) - \psi(r_{j+1})))\Bigg|. \end{aligned} \tag{5.18}$$

We estimate the first term on the right-hand side of (5.18) independently of $k$ as

$$\sum_{j=n}^{\infty}(|\psi_k(r_j) - \psi_k(r_{j+1})| + |\psi(r_j) - \psi(r_{j+1})|) = \psi_k(r_n) + \psi(r_n) \leqslant 2\psi(2r_n),$$

and easily conclude that $\lim\limits_{k\to\infty} \sigma_k(t) = \sigma(t)$ for all $t \in [0, T]$.

The results of [2], Theorem 2.2 and Proposition 2.3 state that a pointwise limit of a sequence of regulated functions with uniformly bounded oscillation is regulated. Recall that functions $\sigma_k$ have uniformly bounded oscillation on $[0, T]$ if there exists a function $N$ independent of $k$ such that

$$\operatorname*{Osc}_{\sigma_k,[0,T]}(d) \leqslant N_\sigma(d) \quad \forall\, d > 0, \tag{5.19}$$

where the oscillation $\operatorname*{Osc}\limits_{\sigma_k,[0,T]}(d)$ of $\sigma_k$ on amplitude level $d$ is defined as the maximum of all $n \in \mathbb{N}$ for which there exist pairwise disjoint intervals $(a_i, b_i) \subset [0, T]$, $i = 1, \dots, n$ such that

$$|\sigma(b_i) - \sigma(a_i)| \geqslant d \quad \text{for } i = 1, \dots, n. \tag{5.20}$$

We now check that this condition is satisfied, which will conclude the proof of Lemma 5.3. To this end, we choose a sequence $\{u_l\}_{l\in\mathbb{N}}$ of right-continuous step functions on $[0, T]$ such that

$$\lim_{l\to\infty} \|u_l - u\|_{[0,T]} = 0, \tag{5.21}$$

and for $k, l \in \mathbb{N}$ put

$$\sigma_{kl}(t) = \int_0^\infty \mathfrak{p}_r[u_l](t)\,\mathrm{d}\psi'_k(r). \tag{5.22}$$

It follows from (3.3) that

$$\|\sigma_{kl} - \sigma_k\|_{[0,T]} \leqslant \|u_l - u\|_{[0,T]} \operatorname*{Var}_{[0,2w_0]} \psi'_k. \tag{5.23}$$

To prove that the $\sigma_{kl}$ have uniformly bounded oscillation, we proceed as in [3], Section 2.6. In the rainflow decomposition, each Madelung pair $(\sigma^1_{kl}, \sigma^2_{kl})$ of $\sigma_{kl}$ corresponds to a Madelung pair $(u^1_l, u^2_l)$ of $u_l$, and we have

$$|\sigma^1_{kl} - \sigma^2_{kl}| = 2\psi_k\Big(\frac{|u^1_l - u^2_l|}{2}\Big). \tag{5.24}$$

Similarly, each consecutive pair $(\sigma^1_{kl}, \sigma^2_{kl})$ in the rainflow residual of $\sigma_{kl}$ corresponds to an analogous pair $(u^1_l, u^2_l)$ in the rainflow residual of $u_l$, and thanks to the choice of the initial memory distribution $\lambda_0$ in (2.12), formula (5.24) holds. By [3], Lemma 2.6.16 we have for each $d > 0$ that

$$\operatorname*{Osc}_{\sigma_{kl},[0,T]}(d) = 4M_{\sigma_{kl}}(d) + R_{\sigma_{kl}}(d), \tag{5.25}$$

where $M_{\sigma_{kl}}(d)$ is the number of Madelung pairs of $\sigma_{kl}$ of amplitude larger or equal to $d$, and $R_{\sigma_{kl}}(d)$ is the number of residual pairs of $\sigma_{kl}$ of amplitude larger than or equal to $d$.

Now let $d > 0$ be given. For each Madelung pair and each residual pair of $\sigma_{kl}$ of amplitude larger than or equal to $d$ we associate a Madelung pair or residual pair $(u^1_l, u^2_l)$ of $u_l$, and by virtue of (5.24), they all have the property

$$\psi_k\Big(\frac{|u^1_l - u^2_l|}{2}\Big) \geqslant \frac{d}{2}. \tag{5.26}$$

We have $\psi_k(r) \leqslant \psi(2r)$ for all $r > 0$, hence

$$\psi(|u^1_l - u^2_l|) \geqslant \frac{d}{2}. \tag{5.27}$$

Set $\widehat{d} = \inf\{r > 0; \psi(r) \geqslant d/2\} > 0$. Then $|u^1_l - u^2_l| \geqslant \widehat{d}$ for each $l \in \mathbb{N}$ and each Madelung pair and each residual pair of $u_l$. Since the sequence $\{u_l\}$ is uniformly convergent, it has uniformly bounded oscillation and $\operatorname*{Osc}_{u_l,[0,T]}(\widehat{d}) \leqslant N_u(\widehat{d})$, and consequently $\operatorname*{Osc}_{\sigma_{kl},[0,T]}(d) \leqslant N_u(\widehat{d})$ independently of $k$ and $l$. It follows from the uniform convergence in (5.23) that $\sigma_k$ have uniformly bounded oscillation, hence $\sigma$ is regulated, which we wanted to prove. $\square$

**Lemma 5.4.** *Let the hypotheses of Proposition* 5.1 *hold. Then the function* $\sigma$ *given by* (5.7) *is right-continuous.*

Proof. We proceed by contradiction. Assume that there exists $t$ such that $\limsup_{\tau \searrow t} |\sigma(\tau) - \sigma(t)| = 2\eta > 0$. We choose a sequence $t_k \searrow t$ and a number $r_* > 0$ such that

$$\lim_{k\to\infty} \left| \int_{r_*}^{\infty} \left( \frac{\partial^-}{\partial r} \mathfrak{p}_r[u](t_k) - \frac{\partial^-}{\partial r} \mathfrak{p}_r[u](t) \right) \mathrm{d}\psi(r) \right| > \eta. \tag{5.28}$$

Let $\{r_j\}$ be the memory sequence of $u$ at time $t$ as in Lemma 5.2. We choose $r_* \in (r_{n+1}, r_n)$, and assume for definiteness that $n$ is odd, that is, by virtue of (5.10),

$$\frac{\partial^-}{\partial r} \mathfrak{p}_r[u](t) = 1 \quad \text{for } r \in (r_{n+1}, r_n]. \tag{5.29}$$

In other words, there exists $\underline{u} \in \mathbb{R}$ such that

$$\mathfrak{p}_r[u](t) = \underline{u} + r \quad \text{for } r \in [r_{n+1}, r_n]. \tag{5.30}$$

From the inequalities $u(t) - r_{n+1} \leqslant \mathfrak{p}_{r_{n+1}}[u](t) \leqslant u(t) + r_{n+1}$ and from (5.30) it follows that

$$\underline{u} \leqslant u(t) \leqslant \underline{u} + 2r_{n+1}. \tag{5.31}$$

We further fix some $r^* \in (r_n, r_{n-1})$ and $k \in \mathbb{N}$ such that

$$\delta := \sup_{\tau \in [t, t_k]} |u(\tau) - u(t)| < \min\{r_* - r_{n+1}, r^* - r_n\}. \tag{5.32}$$

For $\tau \in [t, t_k]$ we have by virtue of (3.4) and (5.30)–(5.31) that

$$u(\tau) - \mathfrak{p}_{r_*}[u](\tau) \leqslant u(t) - \mathfrak{p}_{r_*}[u](t) + 2\delta \leqslant 2r_{n+1} - r_* + 2\delta,$$

hence

$$u(\tau) - \mathfrak{p}_{r_*}[u](\tau) - r_* \leqslant 2(\delta - (r_* - r_{n+1})) =: -\omega < 0.$$

It follows from Lemma 3.1 that

$$\mathfrak{p}_r[u](\tau) \leqslant \mathfrak{p}_r[u](t) \quad \forall\, \tau \in [t, t_k],\ \forall\, r \geqslant r_*. \tag{5.33}$$

On the other hand, we have $\mathfrak{p}_{r^*}[u](t) = \underline{u} + 2r_n - r^*$ by (5.8), and a similar argument as above yields

$$u(\tau) - \mathfrak{p}_{r^*}[u](\tau) + r^* \geqslant u(t) - \mathfrak{p}_{r^*}[u](t) + r^* - 2\delta \geqslant 2(r^* - r_n - \delta) =: \omega > 0,$$

and by Lemma 3.1 we have

$$\mathfrak{p}_r[u](\tau) \geqslant \mathfrak{p}_r[u](t) \quad \forall\, \tau \in [t, t_k],\ \forall\, r \geqslant r^*. \tag{5.34}$$

In particular, by (5.33)–(5.34),

$$\mathfrak{p}_r[u](\tau) = \mathfrak{p}_r[u](t) \quad \forall\, \tau \in [t, t_k],\ \forall\, r \geqslant r^*. \tag{5.35}$$

We now distinguish two cases.

**A**: $u(\tau) \geqslant \underline{u}\ \forall\, \tau \in [t, t_k]$.

By [14], Lemma 4.1 we have for all $r > 0$ that

$$\mathfrak{p}_r[u](t) = \max\{u(t) - r, \min\{\mathfrak{p}_r[u](t-), u(t) + r\}\}. \tag{5.36}$$

By hypothesis, we have for $r \in [r_{n+1}, r_n]$ that $\mathfrak{p}_r[u](t) = \underline{u}+r$. Hence, either $u(t) = \underline{u}$ and $\mathfrak{p}_r[u](t-) \geqslant \underline{u} + r$, or $u(t) > \underline{u}$ and $\mathfrak{p}_r[u](t-) = \underline{u} + r$.

We define for $\tau \in [0, t_k]$ an auxiliary function

$$u^\flat(\tau) = \begin{cases} u(\tau) & \text{for } \tau \in [0, t), \\ \underline{u} & \text{for } \tau \in [t, t_k]. \end{cases} \tag{5.37}$$

Then, still by [14], Lemma 4.1 we have $\mathfrak{p}_r[u^\flat](t) = \min\{\mathfrak{p}_r[u](t-), \underline{u} + r\} = \underline{u} + r$ for all $r \in [0, r_n]$. By Lemma 3.2, we have

$$\mathfrak{p}_r[u](\tau) \geqslant \mathfrak{p}_r[u^\flat](\tau) \quad \text{for } r \in [0, r_n] \text{ and } \tau \in [t, t_k]. \tag{5.38}$$

Comparing (5.38) with (5.35), (5.33), and (5.30) we obtain that

$$\mathfrak{p}_r[u](\tau) = \mathfrak{p}_r[u](t) \quad \forall\, \tau \in [t, t_k], \forall\, r \in [r_*, r_n] \cup [r^*, \infty). \tag{5.39}$$

We have in particular $\mathfrak{p}_{r_n}[u](\tau) = \underline{u} + r_n$ by (5.30) and $\mathfrak{p}_{r^*}[u](\tau) = \underline{u} + 2r_n - r^*$ by (5.9), hence

$$\mathfrak{p}_{r^*}[u](\tau) - \mathfrak{p}_{r_n}[u](\tau) = r_n - r^*.$$

Since $|(\partial^-/\partial r)\mathfrak{p}_r[u](\tau)| \leqslant 1$ for all $r > 0$, we necessarily have

$$\frac{\partial^-}{\partial r}\mathfrak{p}_r[u](\tau) = -1 = \frac{\partial^-}{\partial r}\mathfrak{p}_r[u](t) \quad \forall\, r \in (r_n, r^*],$$

which, together with (5.39), is in contradiction with (5.28).

**B**: $\forall\, k \in \mathbb{N}\ \exists \tau_k \in [t, t_k]\colon\ u(\tau_k) < \underline{u}$.

The right-continuity of $u$ implies that $u(t) = \underline{u}$ and (5.30) holds for $r \in [0, r_n]$, that is, $r_{n+1} = 0$. Set $u_k = \inf\{u(\tau)\colon\ \tau \in [t, t_k]\}$, and

$$u^\flat(\tau) = \begin{cases} u(\tau) & \text{for } \tau \in [0, t), \\ u_k & \text{for } \tau \in [t, t_k]. \end{cases} \tag{5.40}$$

From (5.36) it follows that $\mathfrak{p}_r[u](t-) \geqslant \underline{u} + r$ for $r \in [0, r_n]$ and $\mathfrak{p}_r[u](t-) = \mathfrak{p}_r[u](t)$ for $r \geqslant r_n$. We have again for $\tau \in [t, t_k]$ that

$$\mathfrak{p}_r[u^\flat](\tau) = \min\{\mathfrak{p}_r[u](t-), u_k + r\} = \begin{cases} u_k + r & \text{for } r \in [0, r_k^*], \\ \mathfrak{p}_r[u](t) & \text{for } r > r_k^*, \end{cases} \tag{5.41}$$

where

$$r_k^* = r_n + \frac{1}{2}(\underline{u} - u_k). \tag{5.42}$$

By Lemma 3.2 we have for $\tau \in [t, t_k]$ and for all $r > 0$ that

$$\mathfrak{p}_r[u](\tau) \geqslant \mathfrak{p}_r[u^\flat](\tau). \tag{5.43}$$

There are still two cases to distinguish:

**B1**: $u(t_k) = u_k$ for infinitely many indices $k$.

Then

$$\mathfrak{p}_r[u](t_k) = \begin{cases} u_k + r & \text{for } r \in [0, r_k^*], \\ \mathfrak{p}_r[u](t) & \text{for } r > r_k^*, \end{cases} \tag{5.44}$$

so that
$$\frac{\partial^-}{\partial r}\mathfrak{p}_r[u](t_k) = \frac{\partial^-}{\partial r}\mathfrak{p}_r[u](t) \quad \forall\, r \in [0, r_n] \cup [r_k^*, \infty),$$
and
$$\frac{\partial^-}{\partial r}\mathfrak{p}_r[u](t_k) - \frac{\partial^-}{\partial r}\mathfrak{p}_r[u](t) = 2$$
in $(r_n, r_k^*]$. Hence,

$$\begin{aligned} \int_{r_*}^\infty \Big(\frac{\partial^-}{\partial r}\mathfrak{p}_r[u](t_k) - \frac{\partial^-}{\partial r}\mathfrak{p}_r[u](t)\Big)\,\mathrm{d}\psi(r) &= \int_{r_*}^\infty 2\chi_{(r_n, r_k^*]}(r)\,\mathrm{d}\psi(r) \\ &= 2(\psi(r_k^*) - \psi(r_n)). \end{aligned} \tag{5.45}$$

We have $\lim_{k\to\infty} r_k^* = r_n$ by (5.42). Since $\psi$ is right-continuous, the right-hand side of (5.45) tends to 0 as $k \to \infty$, which contradicts the hypothesis (5.28).

**B2**: $u(t_k) > u_k \ \forall\, k > k_0$.

For each $k > k_0$ we find a sequence $\{\tau_i\}$ in $(t, t_k)$ such that $u(\tau_i) \searrow u_k$ as $i \to \infty$. We have for all $r > 0$ the inequality $\mathfrak{p}_r[u](\tau_i) \leqslant u(\tau_i) + r$ by definition of the play, and $\mathfrak{p}_r[u](\tau_i) \geqslant \mathfrak{p}_r[u^\flat](\tau_i) = u_k + r$ by (5.43). For $\tau \in [\tau_i, t_k]$ we have

$$\begin{aligned} u(\tau) - \mathfrak{p}_{r_*}[u](\tau) - r_* &\leqslant u(\tau_i) - \mathfrak{p}_{r_*}[u](\tau_i) - r_* + 2 \sup_{s\in[\tau_i,\tau]} |u(s) - u(\tau_i)| \\ &\leqslant u(\tau_i) - u_k - 2r_* + 2\delta < 0 \end{aligned}$$

and we may use Lemma 3.1 to conclude that $\mathfrak{p}_r[u](t_k) \leqslant \mathfrak{p}_r[u](\tau_i)$ for $r \geqslant r_*$. Letting $i \to \infty$ we obtain $\mathfrak{p}_r[u](t_k) = u_k + r$ for $r \in [0, r_k^*]$ and we argue as in the case B1 to contradict the inequality (5.28). This completes the proof for the case that $n$ is even. For $n$ odd, the argument is similar. □

We now easily finish the proof of Proposition 5.1. It suffices to combine the three Lemmas 5.2, 5.3, 5.4.

We conclude this section by recalling Prandtl-Ishlinskii superposition and inversion formulas proved in [12], Corollaries 3.3, 3.4.

**Proposition 5.5.** *Let $u \in G_R[0,T]$ be given and let $\varphi, \psi$ be nondecreasing right-continuous functions, $\varphi(0) = \psi(0) = 0$. For $t \in [0,T]$ put $v(t) := \mathcal{P}_\varphi[u](t)$. Then we have*

$$\mathcal{P}_\psi[\lambda_\varphi, v] = \mathcal{P}_{\psi\circ\varphi}[u], \tag{5.46}$$

*with initial condition*

$$\lambda_\varphi(r) = \min\{-\varphi(2w_0) + r, 0\} \tag{5.47}$$

*analogous to* (2.12).

**Corollary 5.6.** *Let $\psi$ be as in Proposition* 5.5 *and let the equation*

$$\varphi(r) = \psi(\varphi(r)) + r \quad \forall\, r \geqslant 0 \tag{5.48}$$

*admit a nondecreasing right-continuous solution $\varphi$. For $v \in G_R[0,T]$ set $w = \mathcal{P}_\varphi[u]$. Then*

$$w(t) = \mathcal{P}_\psi[\lambda_\varphi, w](t) + v(t) \tag{5.49}$$

*for all $t \in [0,T]$.*

Corollary 5.6 deals precisely with the situation in equation (2.17), where $w$ is replaced with $2w$, $v$ with $2v$, and $\psi$ with $2\kappa\psi$.

## 6. An application in financial markets

In the situation of Corollary 5.6, we can formally define $\varphi$ as the inverse mapping $(I-\psi)^{-1}$ to $I-\psi$, where $I$ is the identity. However, if $I-\psi$ is not monotone, the inverse is not uniquely defined. Figures 1–2 illustrate the possibilities that can happen if the function $x \mapsto x - \psi(x)$ does not admit a continuous inverse. In view of the definition of the market sentiment in (2.14), singular behavior is to be expected whenever classes $A_r$ of traders in some interval $[r_1, r_2]$ have too big an influence on the overall market sentiment. This will then initiate a cascade.

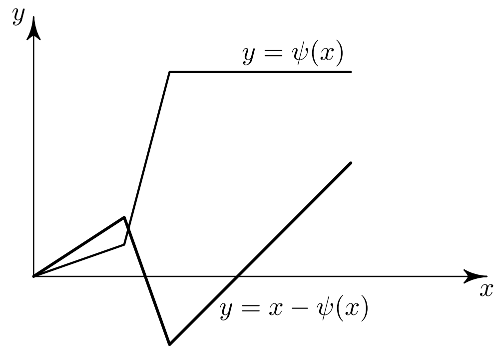


Figure 1. Primary response curves of $\mathcal{P}_\psi$ and $I - \mathcal{P}_\psi$.

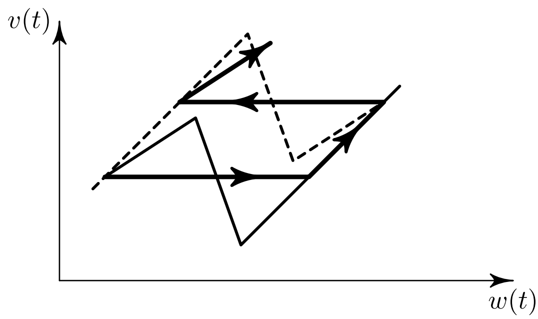


Figure 2. A financial crash.

In this case, there exists a continuum of nondecreasing solutions $\varphi(r)$ of the equation $\varphi(r) = \psi(\varphi(r)) + r$, which generate different Prandtl-Ishlinskii operators with nondecreasing primary response curves according to Corollary 5.6. Consider an increasing input $v(t)$. Then equation (5.49) admits an increasing (possibly discontinuous even if $v$ is continuous) solution $w$. However, it admits singular solutions $w$ as well, with downward jumps (a financial crash!) represented on Figure 2. The full line is the ascending branch of the operator $I - \mathcal{P}_\psi$, the dashed line is the descending branch, and the bold path with arrows is the trajectory of a singular solution of equation (2.17).

## Appendix: Approximation of right-continuous functions

**Proposition A.1.** *Let $\psi\colon [0,\infty) \to [0,\infty)$ be a right-continuous nondecreasing function, $\psi(0) = 0$, and let $\{\delta_k\}_{k\in\mathbb{N}}$ be a sequence in $(0,1]$ such that $\lim\limits_{k\to\infty} \delta_k = 0$, $k\delta_k \geqslant 1$, $\lim\limits_{k\to\infty} k\delta_k = +\infty$. Let $\psi_k$ for $k \in \mathbb{N}$ be the solution of the equation*

$$\frac{1}{k}\psi_k'(r) + \psi_k(r) = \psi((1+\delta_k)r), \quad \psi_k(0) = 0. \tag{A.1}$$

*Then $\psi_k$ are smooth, nondecreasing, and $\lim\limits_{k\to\infty} \psi_k(r) = \psi(r)$ for every $r \geqslant 0$.*

P r o o f. We have the explicit formula

$$\psi_k(r) = k\int_0^r \mathrm{e}^{k(\varrho - r)}\psi((1+\delta_k)\varrho)\,\mathrm{d}\varrho. \tag{A.2}$$

Since $\psi$ is nondecreasing, we have

$$\psi_k(r) \leqslant \psi((1+\delta_k)r)k\int_0^r \mathrm{e}^{k(\varrho - r)}\,\mathrm{d}\varrho = (1 - \mathrm{e}^{-kr})\psi((1+\delta_k)r),$$

so that $\psi_k'(r)/k \geqslant \mathrm{e}^{-kr}\psi((1+\delta_k)r) \geqslant 0$ by virtue of (A.1).

We now fix $R > 0$ and prove that $\psi_k(r) \to \psi(r)$ for each $r \in [0,R]$. On $[0,R]$, the function $\psi$ can be represented by the sum

$$\psi(r) = \psi^0(r) + \sum_{i=1}^{\infty} \alpha_i \psi^i(r), \tag{A.3}$$

where $\psi_0$ is continuous and nondecreasing,

$$\psi^i(r) = \begin{cases} 1 & \text{for } r \geqslant s_i, \\ 0 & \text{for } r \in [0, s_i), \end{cases} \qquad \text{for } i \in \mathbb{N}.$$

The set $\{s_i\colon\ i \in \mathbb{N}\} \subset (0,R]$ contains all discontinuity points of $\psi\big|_{[0,R]}$, $\alpha_i \geqslant 0$ for all $n \in \mathbb{N}$, and $\sum\limits_{i=1}^{\infty} \alpha_i < \infty$. We define the sequence of functions $\{\psi_k^i\}$ as solutions of the equations

$$\frac{1}{k}(\psi_k^i)'(r) + \psi_k^i(r) = \psi^i((1+\delta_k)r), \quad \psi_k^i(0) = 0. \tag{A.4}$$

It is easy to see that

$$\psi_k^0 \to \psi^0 \quad \text{uniformly on } [0,R]. \tag{A.5}$$

Indeed, by formula (A.2) we have for $r \in [0,R]$ that

$$\psi^0(r) - \psi_k^0(r) = \mathrm{e}^{-kr}\psi^0(r) + k\int_0^r \mathrm{e}^{k(\varrho - r)}(\psi^0(r) - \psi^0((1+\delta_k)r))\,\mathrm{d}\varrho. \tag{A.6}$$

Let $\varepsilon > 0$ be given. We find $\omega > 0$ such that $\psi^0(r) < \varepsilon/4$ in $[0,\omega]$, and $k_0 \in \mathbb{N}$ such that $\mathrm{e}^{-k\omega}|\psi^0|_{[0,R]} < \varepsilon/4$ for $k > k_0$. Then the first term on the right-hand side of (A.6) satisfies

$$\mathrm{e}^{-kr}\psi^0(r) < \frac{\varepsilon}{2} \quad \text{for } k > k_0. \tag{A.7}$$

By substituting $\varrho = r - z/k$ we rewrite the integral term on the right-hand side of (A.6) as

$$\int_0^{kr} \mathrm{e}^{-z}\Big(\psi^0(r) - \psi^0\Big(r + \delta_k r - (1+\delta_k)\frac{z}{k}\Big)\Big)\,\mathrm{d}z.$$

We split this integral into two parts:

$$\begin{aligned} I_k^1 &= \int_{r/\delta_k}^{kr} \mathrm{e}^{-z}\Big(\psi^0(r) - \psi^0\Big(r + \delta_k r - (1+\delta_k)\frac{z}{k}\Big)\Big)\,\mathrm{d}z, \\ I_k^2 &= \int_0^{r/\delta_k} \mathrm{e}^{-z}\Big(\psi^0(r) - \psi^0\Big(r + \delta_k r - (1+\delta_k)\frac{z}{k}\Big)\Big)\,\mathrm{d}z. \end{aligned}$$

We have indeed

$$\begin{gathered} |I_k^1| \leqslant 2\mathrm{e}^{-r/\delta_k}|\psi^0|_{[0,(1+\delta_k)r]}, \\ |I_k^2| \leqslant \max_{|\hat\delta| \leqslant \hat\delta_k} |\psi^0(r) - \psi^0(r+\hat\delta)|, \quad \widehat{\delta}_k = R\Big(\delta_k + \frac{1}{k\delta_k} + \frac{1}{k}\Big). \end{gathered}$$

We now find $k_1 \in \mathbb{N}$ such that for $k > k_1$ we have $|I_k^1| + |I_k^2| < \varepsilon/2$, and (A.5) follows.

The proof of the convergence $\psi_k^i(r) \to \psi^i(r)$ for $i \geqslant 1$ is straightforward. We have by (A.2) that

$$\psi_k^i(r) = \begin{cases} 0 & \text{for } r < \dfrac{s_i}{1+\delta_k}, \\[2ex] 1 - \mathrm{e}^{k(s_i/(1+\delta_k) - r)} & \text{for } r \geqslant \dfrac{s_i}{1+\delta_k}. \end{cases} \tag{A.8}$$

In particular, we have $\psi_k^i(s_i) = 1 - \mathrm{e}^{-k\delta_k s_i/(1+\delta_k)}$, and we easily check that $\psi_k^i(r) \to \psi^i(r)$ as $k \to \infty$ for all $r \in [0, R]$.

The above argument shows that putting for $n \in \mathbb{N}$

$$\psi^{(n)}(r) = \psi^0(r) + \sum_{i=1}^{n} \alpha_i \psi^i(r)$$

and denoting by $\psi_k^{(n)}$ the solution of (A.1) corresponding to $\psi^{(n)}$, we have $\psi_k^{(n)}(r) \to \psi^{(n)}(r)$ as $k \to \infty$ for all $r \in [0, R]$. Since the convergences $\psi_k^{(n)} \to \psi_k$ and $\psi^{(n)} \to \psi$ as $n \to \infty$ are uniform independently of $k$, we obtain the assertion. $\square$

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