Rational and Boundedly Rational Inaction in a Macroeconomic Model

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Abstract

Inaction and stickiness in economic and financial systems can arise in various ways. It can be rational, such as in the presence of activity costs, or may be caused by the bounded rationality of agents. Unsurprisingly, this has resulted in various modeling approaches with differing justifications at the micro and aggregate levels and degrees of analytical tractability.

Here we consider the situation where a model variable only changes so as to maintain a maximal allowable difference from a related variable. This form of inaction/stickiness naturally leads to interesting consequences such as path-dependence, a continuum of feasible equilibria, and highly non-trivial responses to shocks of varying magnitudes — even in very simple otherwise-linear systems.

We show how all these phenomena can be rigorously analyzed using mathematical play operators. In an economic setting these can be interpreted in two very different ways, either as the optimal response to a variational problem that minimizes total switching costs (within a given interval of admissible values), or as a (non-forward-looking) form of bounded rationality.

We choose a simple, well-understood, unique equilibrium macroeconomic model as our starting point and then introduce a play operator in two ways. First we imbue the aggregate inflation expectation term with this form of boundedly rational inaction. In the second model we instead use a play operator to model interest rate setting by the Central Bank.

Typically, after sufficiently small shocks the system will revert to its prior (path-dependent) equilibrium but larger shocks will permanently change the equilibrium value. Furthermore, a stability analysis shows that the path to this new equilibrium may be very long with a highly unpredictable, sometimes counter-intuitive, endpoint. Indeed at certain model parameters exogenously-triggered runaway inflation can occur. We also observe an additional potentially destabilizing effect due to Central Bank inaction. Finally we show how multiple play operators may be incorporated to form more sophisticated models.

Keywords: bounded rationality, stickiness, mathematical models, adaptive expectations, path-dependence, sticky inflation.

1. Introduction

The temporary inaction of certain agents is an important qualitative feature of much real-world economic activity. However, just as in engineering/physical systems with frictions, its inclusion in quantitative mathematical models is non-trivial — with the additional complication that in economics there may be multiple causes of 'friction' within a heterogeneous population.

In this paper we study *play operators* in an economic setting. These operators are widely-used in engineering, and have already been applied to both financial modeling [1] and microeconomics [2]. However, here we shall focus on macroeconomics.

For this initial investigation we introduce play operators into a toy Dynamic Stochastic General Equilibrium (DSGE) model in two different places. The first corresponds to a simple model of boundedly rational stickiness in (aggregate) inflation expectations. Play operators have an extremely useful and unusual composition/aggregation property which holds even when agents are connected on an arbitrary network and are influenced both by the actual inflation rate and the expectations of their neighbors. This is discussed in Section 1.6 but the analysis presented here does not make use of it and in this sense is preliminary. The second use of a play operator does not require any aggregation and represents a particular form of rational/strategic inaction in the setting of interest rates by the Central Bank.

Of course no complex economic 'entity', such as an aggregated Representative Agent or a committee of Central Bankers, is likely to be described more than approximately by a simple mathematical operator. However the very different, but quite intuitive, economic interpretations of play operators together with their analytical tractability helps to justify studying models that include them.

Both the analysis and the numerical simulations demonstrate interesting dynamics in the presence of noise and exogenous shocks that have plausible real-world interpretations and thus potential explanatory power. It should be noted that the main qualitative features appear to be robust and not due to the simplicity of the underlying toy model.

We start by introducing the mathematical definition of the play operator (and its dual the stop operator) in an abstract setting and stating the minimization problem that it solves.

1.1. Play and stop operators

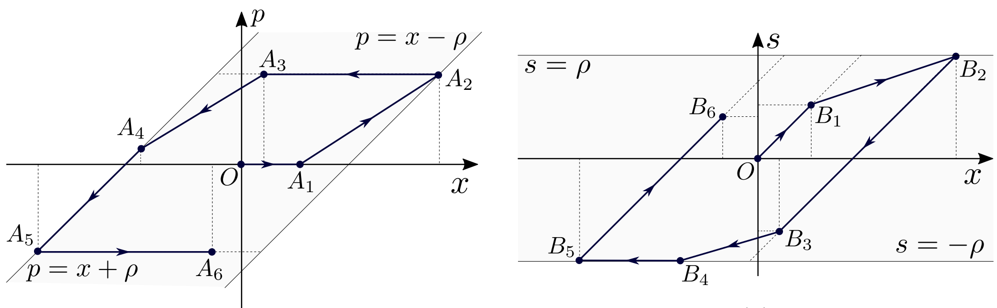
We assume the following rules that define how the output p_t of a play operator varies with the input x_t at integer times t:

(i) The value of the difference $|p_t - x_t|$ never exceeds a certain bound ρ;

(ii) As long as the above restriction is satisfied, the output does not change, i.e. $|x_t - p_{t-1}| \leq \rho$ implies $p_t = p_{t-1}$;

(iii) If the output has to change, it makes the minimal increment consistent with constraint (i).

Rule (ii) introduces inaction/stickiness into the dependence of p_t on x_t, while (i) states that the output cannot deviate from the input by more than a prescribed threshold value ρ. Hence p_t follows x_t reasonably closely but is 'parsimonious' because it remains indifferent to variations of x_t limited to a (moving) window $p - \rho \le x \le p + \rho$. The last rule (iii) enforces continuity of the relationship between p_t and x_t and, in this sense, can be considered as a technical modeling assumption.



(a) (b)

Figure 1: (a) An illustration of the input-output sequence of the (a) play operator and (b) stop operator. (a) The polyline $OA_1A_2A_3A_4A_5A_6$ represents a sample input-output trajectory for the play operator. The input-output pair (x, p) is bounded to the gray strip between the two parallel lines $p = x \pm \rho$. In [2], this strip is called *band of inactivity*, the line $x = x - \rho$ is called *upward spurt* line while the line $p = x + \rho$ is called *downward spurt* line. The output p remains unchanged for a transition from (x_{t-1}, p_{t-1}) to the next point (x_t, p_t) as long as the pair (x_t, p_{t-1}) fits to the band of inactivity (for example, the transitions from $A_2 = (x_2, p_2)$ to $A_3 = (x_3, p_3)$ with $p_2 = p_3$ or from $A_5 = (x_5, p_5)$ to $A_6 = (x_6, p_6)$ with $p_5 = p_6$). If $x_t > x_{t-1}$ and the point (x_t, p_{t-1}) lies to the right of the inactivity band, then the output increases resulting in the point (x_t, p_t) to lie on the upward spurt curve (for example, the transition from $A_1 = (x_1, p_1)$ to $A_2 = (x_2, p_2)$). Similarly, if $x_t < x_{t-1}$ and the point (x_t, p_{t-1}) lies to the left of the inactivity band, then the output decreases and the point (x_t, p_t) lies on the downward spurt line (for example, the transition from $A_3 = (x_3, p_3)$ to $A_4 = (x_4, p_4)$). (b) The input-output trajectory of the dual stop operator corresponding to the trajectory of the play operator shown in panel (a). Here $s_t = x_t - p_t$; the trajectory is limited to the horizontal strip $-\rho \le s \le \rho$ at all times.

Rules (i)–(iii) are expressed by the formula

$$p_t = x_t + \Phi_\rho(p_{t-1} - x_t) \tag{1}$$

with the piecewise-linear saturation function

$$\Phi_\rho(x) = \begin{cases} \rho & \text{if} \quad x \ge \rho, \\ x & \text{if} \quad -\rho < x < \rho, \\ -\rho & \text{if} \quad x \le -\rho. \end{cases} \tag{2}$$

Relationship (1) is known as the *play* operator with *threshold* ρ, see Fig. 1(a) for a depiction of the operator's behavior. A dual relationship

$$s_t = \Phi_\rho(x_t - x_{t-1} + s_{t-1}) \tag{3}$$

between x_t and the variable

$$s_t = x_t - p_t$$

is referred to as the *stop* operator, see Fig. 1(b). The variable s_t measures the difference between the output and input, hence s_t remains within the bound $|s_t| \le \rho$ at all times.

Interestingly the explicit relationship (1) between pairs of variables has been observed in economic data [2, 3].

One can think of the play operator as having two modes. A 'stuck mode' where it will not respond to small changes in the input and a 'dragged mode' where the absolute difference between the input and output is at the threshold value and any change to the input, in the correct direction, will drag the output along with it.

Equations (1) and (3) will now be denoted by

$$p_t = \mathcal{P}_\rho[x_t], \qquad s_t = x_t - p_t = \mathcal{S}_\rho[x_t], \tag{4}$$

where $\mathcal{P}_\rho$ and $\mathcal{S}_\rho$ are the *play* and *stop* operators with threshold ρ, respectively.

1.2. The play operator as the solution to a variational inequality

The play operator above is in fact the solution to a well-defined variational inequality. So if an agent in an economic model acts as a play operator when responding to some input variable they can, depending upon the context, be considered as engaging in a form of optimizing/minimizing rational behavior.

Given a sequence x_t for $t = 0, \dots, T$ and initial condition $p_0 \in [x_0 - \rho, x_0 + \rho]$ then the play operator p_t is the unique function from $\{0, 1, 2, \dots, T\}$ to $\mathbb{R}$ such that

(a) $|x_t - p_t| \le \rho$ for all $t = 0, \dots, T$ and

(b) for all sequences y_t with $|y_t| \le \rho$,

$$\sum_{s=1}^{t} (x_s - p_s - y_s)(p_s - p_{s-1}) \ge 0 \quad \text{for all} \quad t = 1, \dots, T.$$

Thus over each time step p_t moves as little as possible subject to constraint (a) (without knowledge of future values of the input). This has a very natural interpretation when we make the Central Bank's interest rate policy a play operator in Section 4 — rather than adjust the interest rate r_t at every time step the Bank instead only adjusts r_t subject to conditions (i)-(iii) above where the input is some linear combination of the current inflation rate and output gap.

This optimizing property of play operators was originally stated in continuous-time with absolutely continuous inputs and outputs [4]. However, the above discrete-time formulation is a special case of the results in [5] where the inputs and outputs are piecewise-constant.

1.3. The dynamical consequences of play operators in an otherwise-linear system

A play operator can be in one of two modes. Its output is either 'stuck' at some value or is being 'dragged' along by the input variable because the maximum allowable difference between them has been reached. Each of these modes can be analyzed separately as linear systems using standard stability techniques. However the full 'hybrid' system is nonlinear and displays far richer dynamics in the presence of exogenous noise and shocks (the switching between the two modes and differences in their stability also gives rise to non-Gaussian statistics, see for example [6]).

Here we are able to prove the existence of an entire line interval of feasible equilibrium points, examine their stability, and identify some important consequences of path dependence regarding the effects of exogenous shocks and policy changes upon the state of the system. Furthermore, these effects are plausible in that they both correspond to observed, but potentially puzzling, economic situations and are robust enough to be observed numerically in more sophisticated variants of the model.

The level of mathematical knowledge required to follow most of the arguments is not much more than is needed to examine the existence and stability of equilibria in more traditional, fully linear, models. Also, the threshold, or amount of 'give', in the play operator can be reduced to zero recovering the linear, unique equilibrium, case. Or, to put it another way, we can rigorously show that a plausible, yet analyzable, perturbation (that may be rational or boundedly rational) of a linear model significantly alters the qualitative behavior of the system in recognizable ways. [1]

If the presence of stickiness/inaction/frictions in economics does indeed induce a myriad of coexisting equilibria then phenomena that are not possible (or require a posteriori model adjustments) in unique equilibrium models become not just feasible but inevitable. Perhaps the most obvious of these is *permanence*, also known as remanence, where a system does not revert to its previous state after an exogenous shock is removed. It is of course a central concern of macroeconomics whether or not economies affected by, say, significant negative shocks can be expected to have permanently reduced productivity levels.

For the models studied in this paper, sufficiently small shocks (whether exogenous or applied by policy makers) will not change the equilibrium point and a standard linear stability analysis determines the rate at which the system returns to it. Larger shocks will move the equilibrium point along a line of potential equilibria in the expected direction. But even larger shocks may move the system far enough away from the set of equilibria that the return path and ending point on the interval are very hard to predict. Furthermore, in neither of the last two cases will the system exhibit any tendency to return to its pre-shocked state — the model displays true permanence. A related property is that the model parameters alone cannot determine which equilibrium a system is currently in without knowing important information about the prior states of the system — true path dependence (note that this does not prevent the system from being iterated once the initial conditions are fully specified).

1.4. The play operator as rational inaction

As shown in Section 1.2 a play operator can be interpreted as the optimal solution to a very straightforward variational problem. Thus an agent in an economic model whose output is described by a play operator can be considered as rational, at least within the confines of the model.

In Section 4 we shall model the Interest Rate setting mechanism of the Central Bank by a play operator. The output is of course the rate itself and the input is some linear

[1] In [7] a similar 'stress test' was applied to equilibrium models used in finance. It was shown that even very low levels of irrational or perversely-incentivized rational herding by market participants destabilizes the equilibrium (Brownian motion) solution for an asset price and replaces it with 'boom-and-bust' dynamics that is only evident over long timescales.

combination of output gap and inflation. Rather than change interest rates every time period, the Central Bank does so as infrequently and minimally as possible by satisfying rules (i)–(iii) from Section 1.1.

We note that there are other ways in which inaction can be justified as a rational response. The Rational Inattention literature (see for example [8, 9]) treats economic agents as finite-capacity channels (in Information Theoretic terms) that can only (or choose to only) process new information at a finite rate. They then react optimally in the presence of these constraints but this can involve long delays and periods of non-reaction to changing economic circumstances (see [10] for a DSGE model with rational inattention).

There are also Instantaneous Control Models [11] where a (costly) control mechanism ensures that some quantity never leaves a specified interval by kicking in at the boundaries of the interval. For example, the inventory in a warehouse might be regulated by selling the surplus at a discount when it is full and having to buy at a premium when it is empty. A play operator can indeed fit into this framework but we shall focus upon dynamic rather than stochastic properties.

1.5. The play operator as boundedly rational inaction

Since at least the time of Keynes and his General Theory, the idea that people (as individuals or collectively) will make boundedly rational shortcuts or use 'rules of thumb' has been an important element in certain schools of macroeconomics, especially when the future is highly unpredictable. Various theories of 'Heuristics under Uncertainty' and rules for 'Satisficing' have been observed by experimental economists and formalized by theorists [12, 13].

We can use a play operator with threshold ρ_i to mimic the inflation expectation of a single agent i in the following way. The input to agent i is a combination of both the current inflation rate and the current expectations of those agents who are neighbors of i on some network structure representing relationships within the economy. Agent i's inflation expectation remains fixed/stuck until the difference between their own expectations and the input becomes greater than ρ_i (in either direction) in which case i's expectation moves so as to keep the difference at ρ_i. Note that the operator play combines boundedly rational inertia and anchoring together with a minimal adjustment procedure[2]. The important issue of aggregation into a single Representative Agent will be discussed below.

The research into how expectations are actually formed is extensive but far from conclusive, see for example [14, 15, 16, 17, 18]. However the ideas of threshold effects and a 'harmless interval' of inflation are not new in economics [19, 20, 21, 22, 23] and are consistent with our modeling approach. Also, there is some evidence for this type of inaction in experimental data [2, 3]

The most widely-used models for inaction/stickiness in macroeconomics are 'delayed rationality' approaches such as Calvo pricing [24] and the sticky-information of Mankiw and Reis [25]. Here hypothetical agents instantaneously adjust to the 'correct' rational response

[2]Of course over long timescales any agent would notice eventually that their wages, say, were not keeping up with inflation but we posit that over shorter timescales in the presence of noise this is not a major effect.

but at a fixed rate rather than immediately. This can be represented mathematically by introducing delay terms into the relevant equations. In the absence of noise the same optimal equilibrium solution will be reached as if the stickiness were absent. In macroeconomics delayed-rationality models have the advantage that they do not deviate too far philosophically from the Rational Expectations paradigm [26], are easy to analyze, and often retain the unique-equilibrium property. [3]

However our hypothetical agents are not furnished with any concept of rational expectations, even with a delay. They are truly stuck (not just delayed) until forced to adjust by the magnitude of some discrepancy and they have no awareness of the modeling assumptions. This means that at any moment in time the particular equilibrium being approached is determined by prior states of the system and *not* by modeling assumptions about the future.

Indeed this play operator model of expectations has more in common with approaches that were popular before the rise of Rational Expectations. In particular our agents' expectations are 'backward-looking' as in Adaptive Expectations and some Adaptive Learning [30] models that also generate path-dependent equilibria. However, rather than using, say, lagged inflation values with exponentially-decaying weights, each agent's inflation expectation is determined by a subset of the most recent extrema of their individual inputs and is handled automatically by the play operator as described in Section 1.1.

1.6. The aggregation problem

The standard approach to the problem of aggregating expectations is to introduce a Representative Agent whose expectations are fully-informed, rational and consistent with the model itself. Here, if we suppose that each agent acts like a play operator, then an aggregation of *boundedly* rational agents into a single Representative is required. To examine this issue we shall temporarily generalize the discussion to consider abstract play operators and some of their mathematical properties.

The aggregate of even just two play operators with differing thresholds is no longer a play operator — although the output will still display inactive (as well as less active) modes and there is still a maximum allowable difference between the input and output. This is illustrated in Figure 17(a) in Appendix F for the case of three play operators. However play (and stop) operators are just special cases of a wider class of *Prandtl-Ishlinskii* or PI operators.

PI operators have a remarkable aggregation/composition property. When connected together in an arbitrary network (under mild technical conditions) they collectively act as a *single*, but different, PI operator. Thus, as long as individual agents are represented by PI operators, there is a rigorous aggregation process by which a network of interacting heterogeneous agents can be reduced to a single Representative Agent. This ability to

[3] Continua of equilibria can occur in such models (see [27, 28] and for the special case of passive interest-rate policy see [24, 29]) and is considered an extreme form of *indeterminacy*. This is problematic within a Rational Expectations framework since it makes it harder still to justify how agents' expectations are consistent with the model.

rigorously aggregate non-trivial agents is very unusual — and not just within economics. The result and further references to PI operators can be found in [31]. Furthermore this new composite PI operator can be identified analytically in simple cases or, in general, by measuring the network's response to a monotonic input.

So let us return to the expectations aggregation problem. We can imagine the agents in the economy/model being connected in a way that reflects how much influence expectations (or, say, wage increases) affect their neighbors. Then each agent's individual inflation expectation is modeled by a play operator (or by something 'close' within the space of PI operators) whose input is some combination of both the actual inflation rate and the expectations of their neighbors. By the composition property of PI operators, the aggregate response will also be a (probably quite complicated) PI operator that should nevertheless still have stuck, less active, and more active regimes while limiting the difference between the actual and aggregated expected inflation.

Importantly, when agents are connected into a network structure, internal 'cascades' of changes in expectations can occur. For example, if one especially significant agent suddenly starts to increase their expectation of inflation, this may trigger increases in its network neighbors' expectations and so on. Nevertheless the possibility of such cascades is still captured by the composite PI operator — if cascades can occur then the output of the operator when given a smoothly varying input has discontinuities that correspond to the cascades (see [31]).

Rather than study a sticky expectations model with a single, complicated, aggregate PI operator we choose to leave this for later work. We instead assume that the aggregated PI function is itself just a single play operator — this case can be analyzed in detail and is a necessary first step for a deeper understanding of the dynamics of such systems (although we present simulations for a three-agent model in Appendix F). Also, the space of all possible boundedly rational perturbations to rational models is very large and very hard to study rigorously or even define. This makes the analysis of tractable, plausible, boundedly-rational variants of rational models of independent interest and provides an additional justification for our non-standard but conceptually simple Representative Agent.

1.7. Outline of the paper

We start from a dynamic stochastic general equilibrium (DSGE) macroeconomics model, which includes aggregate demand and aggregate supply equations

$$\begin{aligned} y_t &= y_{t-1} - a(r_t - p_t) + \epsilon_t, \\ x_t &= b_1 p_t + (1 - b_1)x_{t-1} + b_2 y_t + \eta_t \end{aligned} \tag{5}$$

augmented with the rate-setting rule

$$r_t = c_1 x_t + c_2 y_t, \tag{6}$$

where y_t is the output gap (or unemployment rate, or another measure of economic activity such as gross domestic product), x_t is inflation rate, r_t is interest rate, p_t is the economic agents' aggregate expectation of future inflation rate and ϵ_t, η_t are exogenous noise terms.

Time is an integer variable, $t = 1, 2, \ldots$, and the process starts from initial values x_0, y_0, p_0. All the parameters are non-negative and in addition, $b_1 < 1$. This model is close to the starting model used in [6] but simpler in that we do not include the aggregate expectation of the output gap and the correlation between the subsequent values of the interest rate. We also choose to remove the noise term from the interest rate update rule. The inclusion of such factors does not affect our most significant qualitative observations, but would complicate some aspects of the rigorous analysis that we present.

For expository reasons we present the analysis of the inflation expectations model first and in greater detail than for the Central Bank model. In Sections 2.3–2.4 we perform the stability analyses (local and global) for various parameter regimes of the expectations model, with some details relegated to Appendices. The stability properties of the system are not as clear cut as in a truly linear system. In fact, the equations define a piecewise linear (PWL) system, and certain nonlinear effects come into play. In particular, in nonlinear systems an equilibrium may only be *locally* stable. This means that the equilibrium is only stable to perturbations of a certain size — ones that don't move the system outside of a 'basin of attraction' — and this phenomenon is responsible for much of the interesting dynamics in the presence of shocks of differing sizes.

In Sections 3.1–3.6 we present various numerical simulations. We are particularly interested in the transitions between equilibrium states caused by exogenous shocks, and the effects of increasing or decreasing stickiness. Where possible we compare results against the non-sticky model. Permanence is the rule not the exception and there are even parameter regimes where a large enough shock will completely destabilize an apparently stable system via a runaway inflation mechanism. We also compare the statistical output of the model against that of De Grauwe [6] at similar parameters and see the same boom-and-bust cyclicality and heavy-tailed distributions.

In Appendix F we briefly consider a more complicated version of the model with three representative agents all with different inactivity thresholds. This is primarily to demonstrate that multiple play operators can indeed be used together to simulate different representative agents within a model and that the most important qualitative features are unchanged.

In Section 4 we remove the play operator from inflation expectations and add it into the response of the Central Bank instead, as outlined in Section 1.4. We perform a second stability analysis and obtain some interesting new effects — there is the possibility of (quasi)-periodic behavior in the absence of noise and sticky Central Bankers appear to destabilize equilibria.

We conclude with a summary of the main results, some general implications for policy and modeling, and suggestions for future work.

2. Inflation expectations as a play operator

2.1. The model

Equations (5) and (6), completed with formulas (1) and (2), form a closed model for the evolution of the aggregated variables x_t, y_t, r_t, p_t. However, the dependence of these quantities at time t upon their values at time $t-1$ is implicit. In order to implement the model, we proceed by solving equations (5)–(2) with respect to the variables x_t, y_t. As shown in Appendix A, the model can be written in the following equivalent form:

$$z_t = Az_{t-1} + s_t d + N\xi_t \tag{7}$$

where $z_t = (y_t, x_t)^\top$, $\xi_t = (\epsilon_t, \eta_t)^\top$, the superscript $^\top$ denotes transposition, the matrices A, N and the column vector d are defined by

$$A = \frac{1}{\Delta}\begin{pmatrix} 1-b_1 & a(1-b_1)(1-c_1) \\ b_2 & (1-b_1)(1+ac_2) \end{pmatrix}, \quad N = \frac{1}{\Delta}\begin{pmatrix} 1-b_1 & a(1-c_1) \\ b_2 & 1+ac_2 \end{pmatrix}, \tag{8}$$

$$d = \frac{1}{\Delta}\begin{pmatrix} a(b_1c_1-1) \\ -(ab_2+b_1(1+ac_2)) \end{pmatrix}$$

with

$$\Delta = (1-b_1)(1+ac_2) + ab_2(c_1-1) \tag{9}$$

and $s_t = x_t - p_t$ is defined by the equation

$$s_t = \frac{1}{1+\alpha}\,\Phi_{(1+\alpha)\rho}(f_t - f_{t-1} + s_{t-1}) \tag{10}$$

with

$$\alpha = \frac{\Delta}{b_1(1+ac_2)+ab_2}, \tag{11}$$

$$f_t = \frac{\alpha}{\Delta}\big(b_2 y_{t-1} + (1-b_1)(1+ac_2)x_{t-1} + b_2\epsilon_t + (1+ac_2)\eta_t\big). \tag{12}$$

Equations (7), (10) express y_t, x_t and $s_t = x_t - p_t$ explicitly in terms of the previous values of the same variables and the exogenous noise ϵ_t, η_t. We use these equations in all the simulations that follow.

We shall refer to the variable $s_t = x_t - p_t$ as the *perception gap*. Note that (10) defines a stop operator with input f_t and threshold $(1+\alpha)\rho$, which is different from ρ (cf. (2)) and so (10) can be written as

$$s_t = \frac{1}{1+\alpha}\mathcal{S}_{(1+\alpha)\rho}[f_t]$$

using the notation (4). It is important to note that the transition to equations (7), (10) is justified under the condition that α is positive, and we assume this constraint to hold in the rest of the paper. In particular, $\alpha > 0$ whenever $c_1 > 1$ (see Section 2.4).

2.2. An entire line segment of equilibrium points

We begin the analysis of the model (7), (10) by looking at the case of no exogenous noise, i.e. we set $\xi_t = 0$ and consider the equation

$$z_t = Az_{t-1} + s_t d, \qquad z_t = (y_t, x_t)^\top \tag{13}$$

instead of (7) with s_t defined by (10), (11) and

$$f_t = \frac{\alpha}{\Delta}\big(b_2 y_{t-1} + (1-b_1)(1+ac_2)x_{t-1}\big). \tag{14}$$

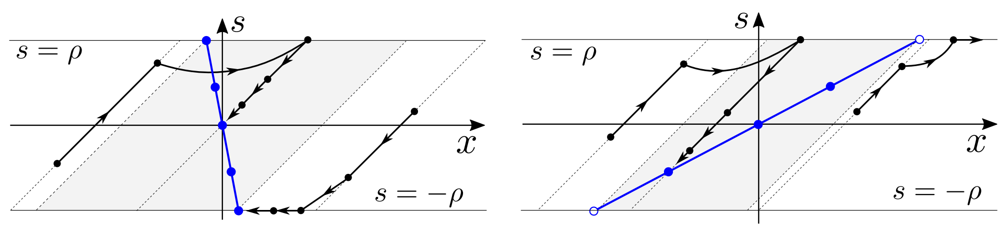
This model has an entire line segment of equilibrium points which corresponds to a continuum of feasible equilibrium states of the economy as a function of the inflation expectations of economic agents. Indeed, equation (13) implies

$$z_* = s_*(\mathbb{I} - A)^{-1} d = s_* \begin{pmatrix} \frac{b_1}{b_2}, & \frac{b_2 + b_1 c_2}{b_2(1-c_1)} \end{pmatrix}^\top \tag{15}$$

for an equilibrium point $z_* = (x_*, y_*)^\top$, where $\mathbb{I}$ is the 2×2 identity matrix. Hence one obtains a different equilibrium for each admissible value of the perception gap variable s_*, i.e. $-\rho \le s_* \le \rho$. Thus, the set of all equilibrium points, which can be denoted as $z_*(s_*)$ for different s_*, can be naturally thought of as a line segment in the phase space of the system, see Fig. 2. In particular, the value of the output gap at an equilibrium, $y_*(s_*)$ ranges over the interval $[-\rho b_1/b_2, \rho b_1/b_2]$ and the equilibrium value of the actual inflation belongs to the range

$$x_*(s_*) = s_* \frac{b_2 + b_1 c_2}{b_2(1 - c_1)} \quad \text{with} \quad -\rho \le s_* \le \rho.$$

Interestingly, at least in this simple model, the range of equilibrium values of the output gap is unaffected by the controls c_1, c_2 applied by the regulator through Taylor's rule (6). However, these controls do affect the range of possible values of the equilibrium inflation rate.



(a) (b)

Figure 2: The projection of the line segment of equilibrium points (blue) onto the (x, s) plane for (a) $c_1 > 1$ and (b) $c_1 < 1$. The segment has a negative slope in (a) and a positive slope in (b). Sample trajectories of system (13) are shown in black.

Equation (15) indicates the difference between the cases $c_1 > 1$ and $c_1 < 1$. When $c_1 > 1$, the equilibrium $z_*(\rho)$ corresponding to the lowest expectation of inflation has the highest value of the output gap and the lowest inflation of all the equilibrium points. Similarly, the equilibrium $z_*(-\rho)$ with the highest expectation of inflation has the lowest value of the output gap and the highest inflation. On the other hand, in case $c_1 < 1$, the equilibrium $z_*(\rho)$ with the highest output gap value has simultaneously the highest inflation rate.

The difference between the cases $c_1 > 1$ and $c_1 < 1$ will be further highlighted in Section 2.4.

2.3. Local stability analysis

System (7), (10) is locally linear in some neighborhood of any equilibrium point from the linear segment (15) with the exception of the two end points $z_*(\pm\rho)$ corresponding to equilibria where the play is right at one end of its inactive band. In other words, for sufficiently small deviations of the vector $z_t = (y_t, x_t)^\top$ from an interior equilibrium $z_*(s_*)$, system (13) is equivalent to

$$z_t - z_*(s_*) = B(z_{t-1} - z_*(s_*)) \tag{16}$$

where

$$B = \begin{pmatrix} \frac{1}{1+a(b_2c_1+c_2)} & \frac{a(b_1-1)c_1}{1+a(b_2c_1+c_2)} \\ \frac{b_2}{1+a(b_2c_1+c_2)} & \frac{(1-b_1)(1+ac_2)}{1+a(b_2c_1+c_2)} \end{pmatrix}$$

As shown in Appendix B, the matrix B is stable for any admissible set of parameter values, hence every equilibrium with $|s_*| < \rho$ is locally stable. This local stability ensures that if a *sufficiently small* perturbation is applied to the system residing at an equilibrium $z_*(s_*)$, removing the perturbation returns the system to the same equilibrium. Further, the eigenvalues of the matrix B determine how quickly (or slowly) the system returns to the equilibrium state. This situation is of course very similar to the expected response in a fully linear equilibrium model. The dependence of the eigenvalues of the parameters of the system is discussed in Appendix C.

However, the situation for these interior equilibria changes markedly for larger perturbations. This is related to the stability properties of the two extreme equilibria $z_*(\pm\rho)$ and is far more subtle as discussed in the next section. In particular, the basin of attraction of the equilibrium decreases and finally vanishes as one approaches either of the extreme equilibrium points along the line segment (15) (the extreme equilibria themselves are stable but not asymptotically stable).

2.4. Global stability analysis

System (13) without stickiness ($\rho = 0$) simply has the form

$$z_t = Az_{t-1}. \tag{17}$$

As shown in Appendix B, its unique zero equilibrium is globally stable if $c_1 > 1$ and is unstable if $c_1 < 1$.

For system (13) with stickiness ($\rho > 0$), equation (17) approximates the dynamics far from equilibrium points because the term s_t in (13) is bounded in absolute value by ρ. In particular, since (17) is unstable for $c_1 < 1$, so is system (13). This creates the possibility of run-away inflation at these values of c_1 (see Section 3.5).

Interestingly, the same condition $c_1 > 1$ that ensures the global stability of system (17), also guarantees the global stability of the set of equilibrium states for the sticky nonlinear system (13). In order to show this, one can use a family of *Lyapunov functions*

$$\begin{aligned} V(x_t, s_t, \nabla_t x, \nabla_t s) &= \tfrac{1}{2}\big(C(\nabla_t x)^2 + G(\nabla_t s)^2 + (Cx_t + Gs_t)^2\big) \\ &+ \gamma\big((Cx_t + Gs_t)\nabla_t x + \tfrac{H}{2C}(Cx_t + Gs_t)^2\big), \end{aligned}$$

where $\nabla_t u = u_t - u_{t-1}$, $u = x, s$. A proper choice of the parameters C, G, H, γ ensures that such a function is non-negative, achieves its minimum zero value on the linear interval of equilibrium states, and decreases to zero along every trajectory of system (13). This allows us to prove that every trajectory of system (13) converges to one of the equilibrium states

(15). In the interest of space, details of the proof are omitted here and will be presented elsewhere.

For system (7) with noise, this global stability result implies that trajectories tend to return towards the segment of equilibrium points after large fluctuations and hover in a vicinity of equilibrium states for extended periods of time. The rate with which the system returns towards the line segment of equilibrium states after a large perturbation is removed is determined by the eigenvalues of the matrix A, see Appendix C.

3. Numerical results

3.1. Parameter values

The default parameter set that we use for numerical simulation is the same as in [6], see Table 1, and we shall explore in detail the surrounding parameter space. Note that,

Parameters	a	b_1	b_2	c_1	c_2
Values	0.2	0.5	0.05	1.5	0.5

Table 1: The set of parameter values.

as an example, if with the above parameters we choose $\rho = \frac{1}{2}$ then the components of the equilibrium points $z_*(s_*) = (y_*(s_*), x_*(s_*))^\top$ range over the intervals

$$y_*(s_*) \in [-5, 5], \qquad x_*(s_*) \in [-6, 6].$$

The choice of ρ is somewhat arbitrary as there is of course no corresponding reference parameter in [6] and so in many of the simulations it will be varied. Also it should be emphasized that these reference parameters are motivated by [6] but very similar numerical results were obtained for other choices.

3.2. Lower inflation volatility due to stickiness

The range of the equilibrium points of the system is directly proportional to the threshold value ρ of the play operator because the perception gap s_* in (15) can take any value in the interval $-\rho \le s_* \le \rho$. In particular, $\rho = 0$ corresponds to the system without stickiness in which the expectation of inflation coincides with the current inflation rate, $p = x$. This system is simply described by the equation

$$z_t = A z_{t-1} + N \xi_t \tag{18}$$

(cf. (7)). In the absence of noise, it has a unique equilibrium at $x = y = 0$.

The sticky system exhibits lower volatility in the inflation rate than the system without stickiness, see Fig. 3. This can be explained by the stability properties of matrices A and B where B is the linearization matrix of (16) for the sticky system at an equilibrium. For the parameter values of Table 1, the spectral radius of the matrix B is smaller than the spectral radius of A (see Appendix C), hence the sticky system tries to revert to equilibrium more strongly within the basin of attraction of individual equilibria, i.e. as long as the perception gap does not become extreme. Fig. 3 shows that the volatility decreases with ρ. For large (compared to ρ) deviations of z_t from the set of equilibrium points, system (7) behaves as (18).

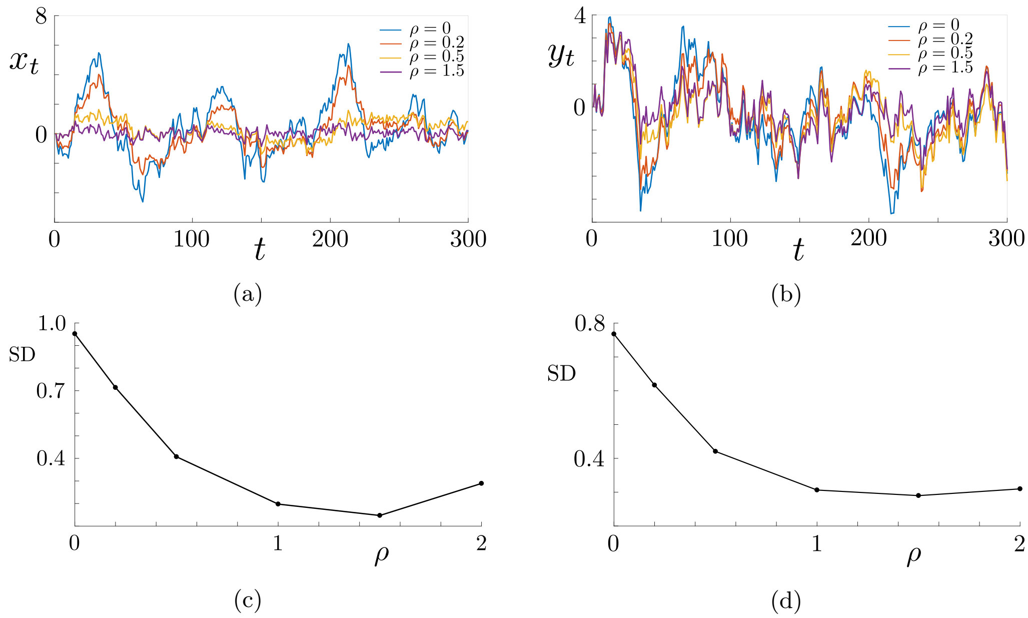


Figure 3: Trajectories of (a) inflation rate x_t and (b) output gap y_t. Measure of volatility of (c) inflation rate and (d) output gap for different values of ρ with standard deviation (SD).

3.3. Transitions between equilibrium states

The system remains within the basin of attraction of a particular equilibrium state $z_*(s_*)$ as long as the perception gap s_t does not reach either of the extreme values $\pm\rho$ and remains confined to the interval $|s_t| < \rho$, see Fig. 4(a,d). But as soon as the perception gap hits the end of its range and starts being 'dragged' by the actual inflation rate (Fig. 4(b,e)) the system transitions to the basin of attraction of a different equilibrium state where s_t becomes 'stuck' again. For this reason, the system stays near equilibrium states which correspond to non-extreme perception gaps for longer periods of time than near extreme ones. Figures 4(c,f) illustrate a transition from the equilibrium state with an extreme perception gap, $z_*(\rho)$, to one with a more moderate perception gap.

3.4. Response to shocks

We shall stress the system by applying supply shocks through the term η_t. The response of the system to demand shocks applied through the term ϵ_t is similar. However, the parameter regime being considered diminishes the effect of relatively small demand shocks due to the small value of $b_2 = 0.05$.

System (18) without stickiness, which has a unique globally stable equilibrium state $x_* = y_* = 0$, as expected returns to the equilibrium (and hovers near it due to noise) after each shock, see Fig. 5(a). Shocks applied to the sticky system (7), (10) result in

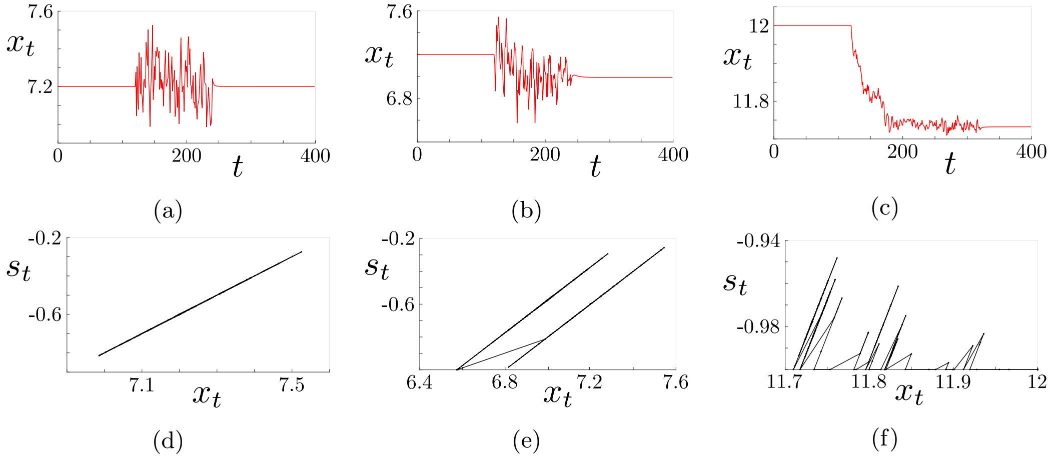


Figure 4: Transitions between the equilibrium states. (a – c) Time traces of inflation rate; (d – f) the corresponding plots in the (x, s)-space exhibiting different transition scenarios. The noise is turned off before and after the interval of time of interest in order to show the equilibrium state at the ends of this interval. (a, d) The perception gap remains within the bounds $|s_t| < \rho$, and the system stays in the basin of attraction of one equilibrium point. The inflation rate $x_*(s_*)$ is the same before and after the noisy interlude. (b, e) The perception gap reaches the extreme value $-\rho$ (the highest expectation of inflation), and the trajectory transits from the basin of attraction of an equilibrium state with higher inflation rate and lower output gap (the right slanted segment in (e)) to the basin of attraction of an equilibrium state with a lower inflation rate and higher output gap (the left slanted segment in (e)). (c, f). A transition from the equilibrium with the highest inflation rate (the rightmost point in (f)) to an equilibrium state with a more moderate inflation rate through the basins of attraction of several other equilibrium states.

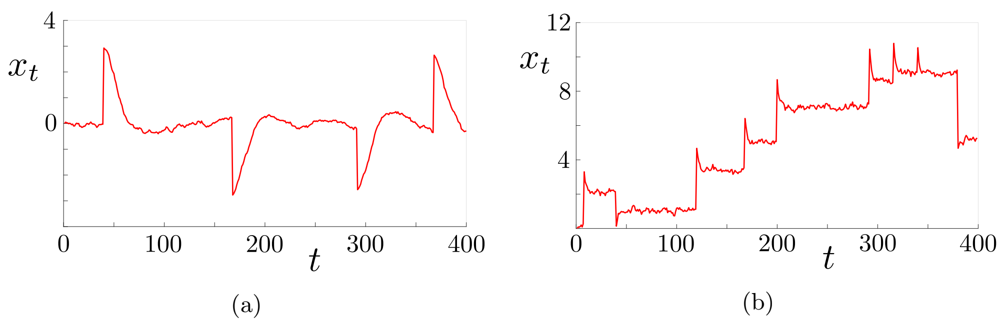
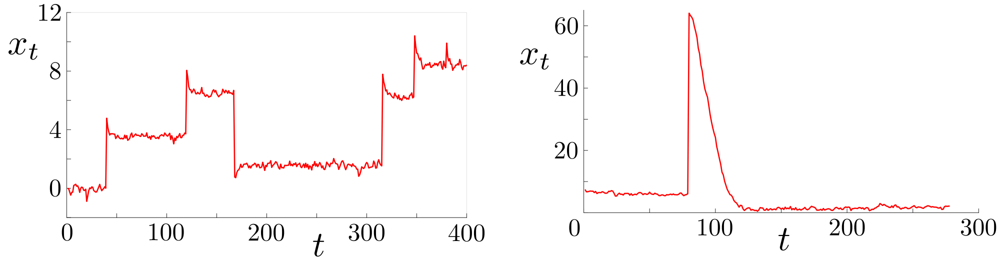


Figure 5: Response to shocks. (a) The system without stickiness ($\rho = 0$) settles to the same unique equilibrium after each shock. (b) The system with stickiness ($\rho = 1$) settles to a new equilibrium after a shock is applied.

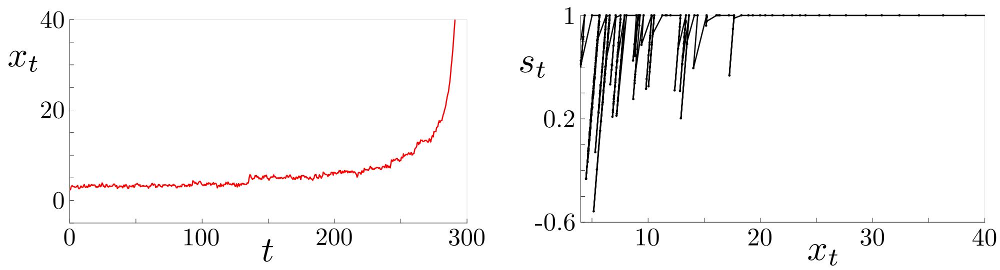
transitions between equilibrium states, see Figure 5(b). Numerical simulation show that shocks of small magnitude typically move the system in the direction of the shock (see Fig. 6(a)). For example, after a shock that pushes up the inflation rate the system settles to a new equilibrium state, which has higher inflation rate (and lower output gap) than the equilibrium occupied prior to the shock. On the other hand, shocks of larger magnitude cause a transition to an equilibrium state that can be hard to predict because such shocks cause a longer and more complex excursion into the phase space far from equilibrium set. In Fig. 6(b), the system resides near an equilibrium with high inflation rate before a shock is applied. Although the shock pushes the inflation even higher, the system eventually settles to an equilibrium with nearly zero inflation rate after the shock is removed.



(a) (b)

Figure 6: Response to shocks of (a) small and (b) large magnitude.

3.5. *The possibility of runaway inflation*



(a) (b)

Figure 7: Run-away inflation scenario. Parameter are $\rho = 1$, $a = 0.3$, $b_1 = 0.5$, $b_2 = 0.05$, $c_1 = 0.9$, $c_2 = 0.01$. The ranges of inflation rate and output gap values at equilibrium states for this set parameter are $x_* \in [-11, 11]$ and $y_* \in [-10, 10]$, respectively. (a) Time series of inflation rate x_t. (b) Trajectory in the (x, s) space.

According to Section 2.4 the system is globally stable for $c_1 > 1$, but becomes unstable for $c_1 < 1$. The latter case creates a possibility of the run-away inflation scenario. It is interesting that as shown in Section 2.3 all the equilibrium points are *locally* stable even if $c_1 < 1$. As a result, dynamics appear to be stable as long as the trajectory is confined to the basin of attraction of an equilibrium state. However, when noise or a shock or another fluctuation drives the trajectory outside this bounded stability domain, the run-away scenario may and is likely to start, see Fig. 7. Just to be clear, the behavior is stable while the perception gap is not extreme, but if a shock causes that to change then the runaway instability can suddenly occur with no change in the system parameters.

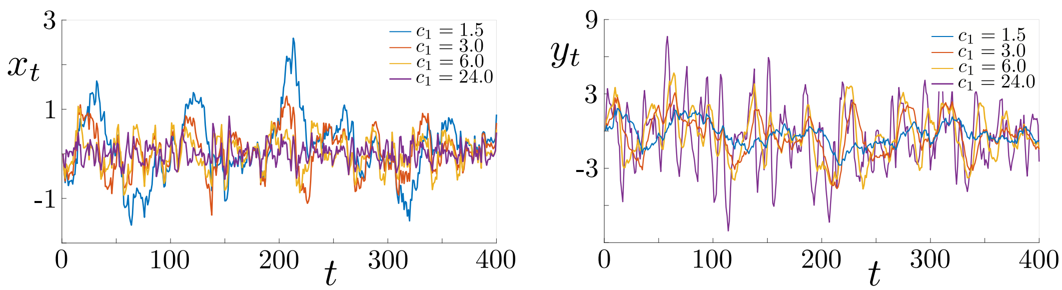
3.6. *A trade-off between inflation and output gap volatility*

Parameters c_1 and c_2 of Taylor's rule (6) control the volatility level of inflation and output gap near an equilibrium state. Numerical simulations of the model with sticky inflation expectation show that when c_1 increases (which corresponds to stronger inflation targeting by the Central Bank), the volatility of the inflation rate decreases, see Fig. 8(a). However, at the same time, the output gap becomes highly volatile with increasing c_1, see Fig. 8(b).

When c_2 increases (stronger output gap targeting), the output gap volatility decreases, see Fig. 9(b). In particular, the case $c_2 = 0$ corresponding to pure inflation targeting in

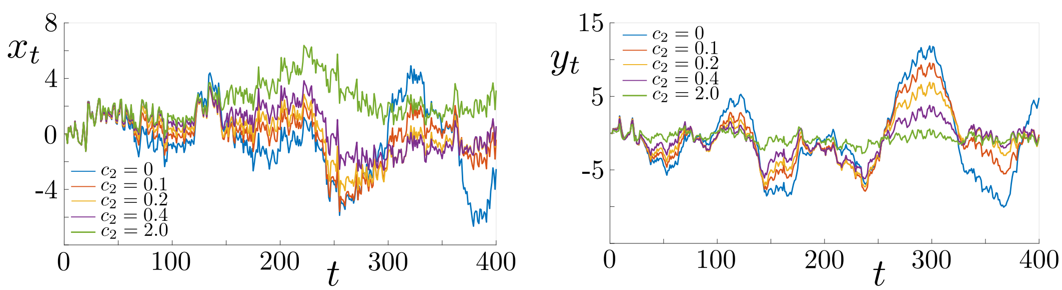
Taylor's rule is characterized by the highest volatility of the output gap. However, from Fig. 9(a), it appears that the inflation rate volatility exhibits a non-monotone behavior with c_2. This is confirmed by Fig. 10, which shows the dependence of the standard deviation of x_t and y_t on c_2 for the trajectories presented in Fig. 9. The inflation rate volatility reaches its minimum for $c_2 \approx 0.8$ for the parameter values a, b_1, b_2, c_1 from Table 1 and $\rho = 1$.

All the above results are in agreement with [6]. In addition, c_1 and c_2 affect the range of the inflation rate value at the equilibrium states for the model (7). According to (15), this range increases with c_2 and decreases with $c_1 - 1$ (for $c_1 > 1$). At the same time, the range of output gap equilibrium values is unaffected by the parameters of Taylor's rule.



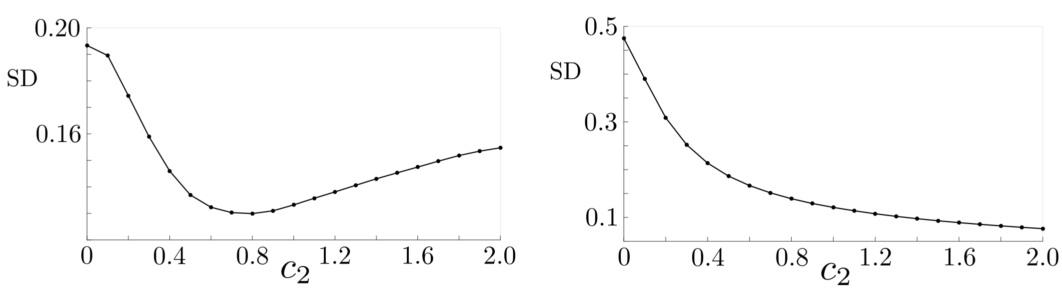
(a) (b)

Figure 8: Numerical simulations of (a) inflation rate, x_t and (b) output gap, y_t for $\rho = 1$ and various values of c_1. The remaining parameters values are from Table 1.



(a) (b)

Figure 9: Numerical simulations of (a) inflation rate, x_t and (b) output gap, y_t for $\rho = 1$ and various values of c_2. The remaining parameter values are from Table 1.



(a) (b)

Figure 10: Measure of the effect of c_2 on volatility of (a) x_t and (b) y_t with standard deviation (SD).

4. The Central Bank as a play operator

The Central Bank policy can be rationally inactive or sticky as noted in Section 1.4. To explore this scenario in this Section we shall replace the Taylor rule (6) with the relation

$$r_t = \mathcal{P}_\sigma[c_1 x_t + c_2 y_t] \tag{19}$$

also involving a play operator. But at the same time, for the sake of simplicity and in order to isolate the effect of stickiness in the Central Bank response upon the system, we remove the play operator from equations (5) thus assuming that the aggregate expectation of inflation equals to the current actual inflation rate, $p_t = x_t$; this corresponds to setting $\rho = 0$ in equations (5). In this case,

$$\begin{aligned} y_t &= y_{t-1} - a(r_t - x_t) + \epsilon_t, \\ x_t &= x_{t-1} + \tfrac{b_2}{1-b_1} y_t + \eta_t. \end{aligned} \tag{20}$$

It would be interesting to consider the model with both sticky inflation expectations and sticky Central Bank response, however this is beyond the scope of this paper.

System (19), (20) can be written in the form (7) with

$$s_t = \mathcal{S}_\sigma[c_1 x_t + c_2 y_t],$$

the matrix A defined by (8), $N = A$, and $d = (a(1-b_1), ab_2)^\top/\Delta$ with Δ defined by (9). The technique presented in Subsection 2.1 can be adapted to convert the implicit system (19), (20) into a well-defined explicit system provided that

$$1 - b_1 - ab_2 > 0. \tag{21}$$

(see Appendix E). Hence, we assume that this condition is satisfied.

Equilibrium states of system (19), (20) with zero noise terms form the line segment

$$(y_*(s_*), x_*(s_*)) = \Big(0, \frac{s_*}{c_1 - 1}\Big), \qquad s_* \in [-\sigma, \sigma]. \tag{22}$$

Notice that the output gap value is zero for all the equilibrium states, while the equilibrium inflation rate ranges over an interval of values. Notably, the local stability analysis (see Appendix E) shows that all the equilibrium states with $s_* \in (-\sigma, \sigma)$ are *unstable* for any set of parameter values. That is, stickiness in the Taylor rule leads to destabilization of equilibrium states.

On the other hand, for large values of $z_t = (y_t, x_t)^\top$, the system can be approximated by equation (17), which is exponentially stable (as shown in Appendix B). This ensures that in the system (19), (20), in the absence of noise, all trajectories converge to a bounded domain Ω surrounding the segment of equilibrium states and, upon entering this domain, remain there. However, since the equilibria are all unstable, more complicated bounded attracting orbits (such as periodic, quasiperiodic, or even chaotic attractors) must occur. Fig. 11 shows a few possibilities for the attractor of system (19), (20) obtained for different sets of parameter values. The attractor belongs to Ω whose size is controlled by the parameter σ of the sticky Taylor rule (19). This size can be estimated using the Lyapunov function introduced in Subsection 2.4.

Finally, we note that in the presence of noise, a trajectory will most likely wander unpredictable around Ω unless kicked outside temporarily by a fluctuation.

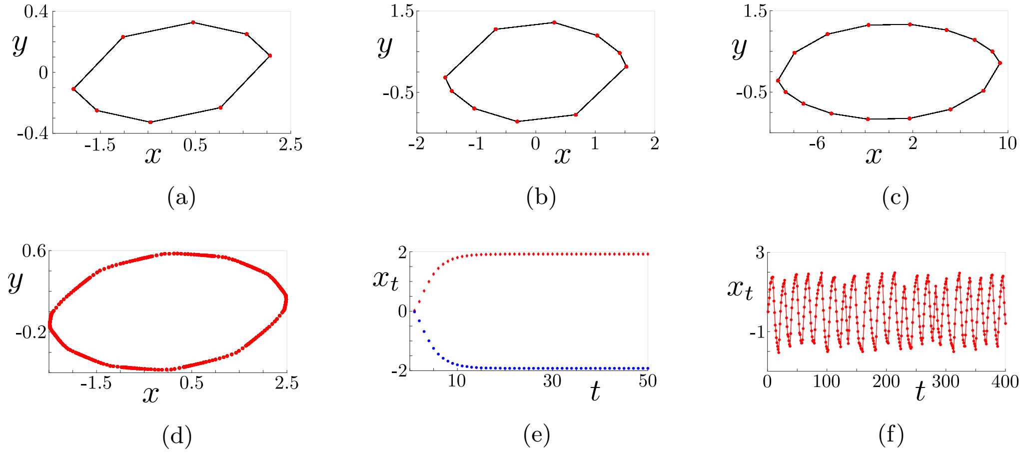


Figure 11: An attractor of system (19), (20) for several parameter sets. (a – c) A periodic orbit (with period 8, 10, 16, respectively) shown on the (x, y) plane for the system without noise. (d) A quasiperiodic orbit. (e) Two equilibrium states corresponding to $s_* = \pm\sigma$ (the time trace of x_t shown for 2 trajectories). (f) Time trace of x_t for a trajectory of the system with noise for the same parameters as in (e).

5. Conclusions

In this paper we have rigorously analyzed simple macroeconomic models using play operators to introduce inaction/stickiness into both inflation expectations and the interest rate setting mechanism of the Central Bank. For such simple models, defined via single (and conceptually quite elementary) changes from a standard one, the play operators introduce surprisingly complicated and subtle-yet-recognizable phenomena into the dynamics.

In the sticky inflation expectations simulations we observed: lower inflation volatility due to stickiness in inflation expectations; permanent transitions to sometimes unexpected equilibrium states due to exogenous shocks; a trade-off between inflation and output gap volatility as the targeting rule is varied, with evidence of cyclicality over long timescales; the possibility of runaway inflation due to exogenous shocks in an apparently stable system; the possibility of cascading effects in more complex models; and strong cyclicality induced by Central Bank stickiness.

Some of the more detailed conclusions of our simulations are specific to the actual models studied but, based upon the mathematics presented here and additional numerical simulations with more complex variants of the models, we believe at least the following qualitative features to be generic and robust.

Firstly, the presence of an entire continuum of feasible equilibria rather than a unique one (or even finite numbers of them). This causes permanence and path-dependence at a deep level. It should be noted that in more sophisticated models, with more sticky agents and variables, the set of possible equilibria may be extremely complicated with the possibility of 'cascades' where one play operator starting to drag causes others to do so. As was outlined in Section 1.6 play operators, when combined appropriately [31], can have a remarkably simple aggregated response — even when connected in a network. This allows for the possibility of (almost)-analytic solutions (using a more general Prandtl-Ishlinskii operator) even when endogenous cascades and rapid transitions between states are occurring and will be the subject of future work.

Secondly, the existence of different modes depending upon whether particular sticky variables are currently stuck or being dragged. After small enough shocks the system will revert to the same equilibrium just as if it were a linear unique equilibrium model. But some modes will be less stable than others (in our sticky expectations model the dragging mode is less stable than the stuck one) and a large enough shock may move the system far enough away from the set of equilibria that the route back to a new (possibly counter-intuitive) equilibrium is both long and unpredictable. An extreme example of this is when the system moves into a completely unstable regime, runaway inflation, without any change in the system parameters.

Our choice of model for this preliminary investigation into treating economic variables as play/stop operators was partly motivated by the work of De Grauwe [6] and Gabaix [32] which used different boundedly rational expectation formation processes in similar DSGE models. However, play operators are also a viable option for modeling other sticky economic variables at both the micro- and macroeconomic levels. To emphasize this, in our second model we used one to represent a sticky (but rational in the sense of solving a certain minimization problem) strategy by the Central Bank. The results suggest that Central Bank inaction/stickiness tends to destabilize equilibria and cause larger fluctuations in the 'Animal Spirits'.

The modeling approach presented above can be considered a 'stress test' of the usual rational expectations assumption in the underlying toy model. Or to put it another way, it is examining the robustness of a modeling assumption rather than just the stability of the solutions within the model. It provides an additional class of simple perturbations to rational models — ones that are genuinely nonlinear and capable of introducing additional phenomena in a way that merely changing the parameters of an equilibrium model cannot.

The most natural extension of this work, as mentioned in Section 1.6 is to replace the single play operator representing inflation expectations of the Representative Agent with a more realistic PI operator based upon either measurements of the actual economy or a network model of the connections between agents in the economy. Once a suitable PI operator has been identified then simulations can be performed almost as easily as with a play operator although identifying the sets of feasible equilibria for example will be more complicated. If one supposes for a moment that such a model displays similar qualitative features and adequately represent an actual economy then there are some significant modeling/policy implications.

There is our original observation that permanence is an inherent property of the system. After sufficiently small shocks the system may return to the same equilibrium but after larger shocks it will not. This does not mean however that the model parameters have changed. Indeed changing the parameters in a unique equilibrium model to match and then try to control a path-dependent reality may well introduce additional instabilities. This would be an interesting line of research.

Furthermore, different path-dependent equilibria have different stability properties and those close to the boundary of the set of feasible equilibria will typically be only marginally stable. So the system's equilibrium may move around the set of feasible equilibria for a very long time, responding proportionately to exogenous stimuli, until suddenly it doesn't! Either a seemingly unremarkable exogenous shock, or a cascade of endogenous sticky quantities changing their mode, takes the system on a far-from-equilibrium (but still bounded) excursion with a very unpredictable outcome somewhere back on the set of feasible equilibria. Trying to control the endpoint while such an event is unfolding, or undoing its consequences afterwards, may be extraordinarily difficult.

Appendix

A. Derivation of equations (7), (10)

Here we show how to obtain equations (7), (10) from model (5)–(2). To this end, we substitute the equation for r_t into the equation for y_t and obtain

$$(1+ac_2)y_t = y_{t-1} - ac_1x_t + ap_t + \epsilon_t.$$

Next, we substitute this equation into the equation for x_t and simplify to obtain

$$\gamma x_t - \beta p_t = b_2 y_{t-1} + (1-b_1)(1+ac_2)x_{t-1} + b_2\epsilon_t + (1+ac_2)\eta_t, \tag{23}$$

where

$$\gamma = 1 + ac_2 + ab_2c_1, \qquad \beta = b_1(1+ac_2) + ab_2.$$

Since $p_t = x_t - s_t$, equation (23) can be rewritten as

$$\alpha x_t + s_t = f_t \tag{24}$$

with α and f_t defined by (11), (14). Therefore, $x_t = \alpha^{-1}(f_t - s_t)$, which combined with (11), (14) gives

$$x_t = \frac{b_2}{\alpha\beta}y_{t-1} + \frac{(1-b_1)(1+ac_2)}{\alpha\beta}x_{t-1} - \frac{1}{\alpha}s_t + \frac{b_2}{\alpha\beta}\epsilon_t + \frac{1+ac_2}{\alpha\beta}\eta_t. \tag{25}$$

Subsequently, substituting equation (25) into equation (5) gives

$$\begin{aligned} y_t \quad = \quad & \frac{ab_2(1-c_1)+\alpha\beta}{\alpha\beta(1+ac_2)}y_{t-1} + \frac{a(1-c_1)(1-b_1)}{\alpha\beta}x_{t-1} \\ & + \frac{a(c_1-1-\alpha)}{\alpha(1+ac_2)}s_t + \frac{\alpha\beta+ab_2(1-c_1)}{\alpha\beta(1+ac_2)}\epsilon_t + \frac{a(1-c_1)}{\alpha\beta}\eta_t. \end{aligned} \tag{26}$$

Equations (25), (26) can be written as system (7) with the matrices A, N and the vector d defined by formulas (8).

Equation (10) can be obtained from relation (24) using the inversion formula for the play operator. This inversion formula is presented for a more general Prandtl-Ishlinskii (PI) operator, including the play operator as a particular case, in Appendix D.

B. Local stability analysis

The characteristic polynomial of matrix B is

$$P_B(\lambda) = \lambda^2 - \lambda\left(\frac{2+ac_2-b_1(1+ac_2)}{1+a(b_2c_1+c_2)}\right) + \frac{1-b_1}{1+a(b_2c_1+c_2)}.$$

Applying Jury's stability criterion to the characteristic polynomial gives the following set of inequalities:

$$\begin{aligned} P_B(1) &= 1 - \frac{2 + ac_2 - b_1(1 + ac_2)}{1 + a(b_2c_1 + c_2)} + \frac{1 - b_1}{1 + a(b_2c_1 + c_2)} > 0, \\ P_B(-1) &= 1 + \frac{2 + ac_2 - b_1(1 + ac_2)}{1 + a(b_2c_1 + c_2)} + \frac{1 - b_1}{1 + a(b_2c_1 + c_2)} > 0, \\ 1 &> \frac{1 - b_1}{1 + a(b_2c_1 + c_2)}. \end{aligned}$$

It is easy to see that all the three inequalities above are satisfied for any set of parameters $a, b_2, c_1, c_2 > 0$ and $0 < b_1 < 1$, hence every equilibrium $z_*(s_*)$ with $|s_*| < \rho$ is locally stable.

Now, let us consider the system without stiction. The characteristic polynomial of matrix A is

$$P_A(\lambda) = \Delta\lambda^2 - (1 - b_1)(2 + ac_2)\lambda + 1 - b_1$$

with Δ defined by (9). Applying Jury's stability criterion, we obtain

$$\begin{aligned} P_A(1) &= 1 - \frac{(1 - b_1)(2 + ac_2)}{\Delta} + \frac{1 - b_1}{\Delta} > 0, \\ P_A(-1) &= 1 + \frac{(1 - b_1)(2 + ac_2)}{\Delta} + \frac{1 - b_1}{\Delta} > 0, \\ 1 &> \frac{1 - b_1}{\Delta}. \end{aligned}$$

Taking into account the constraints $a, b_2, c_1, c_2 > 0$ and $0 < b_1 < 1$, these conditions result in the relationship

$$c_1 > 1.$$

Note that the system $z_t = Az_{t-1}$ is the linearization of sticky system (7) at infinity, hence it describes the return of the sticky system towards near equilibrium dynamics after a large perturbation. Thus, the stability condition $c_1 > 1$ for A agrees with the global stability criterion obtained in Section 2.4.

C. The effect of parameters on stability properties

Here we provide some numerical analysis concerning the effect of the parameters on stability properties of the equilibrium states. Stronger stability generally implies lower volatility and more infrequent transitions between different equilibrium states. We quantify local stability using the maximum absolute value, $|\lambda_{i,e}|$, of eigenvalues of the linearized system at an equilibrium point. The subscripts e and i refer to the system without stickiness ($\rho = 0$) and with stickiness ($\rho = 1$), respectively.

The model contains five other parameters, a, b_1, b_2, c_1 and c_2. Fig. 12 shows the dependence of $|\lambda_{i,e}|$ on the parameter a and implies that the system with stickiness is more stable than the system without stickiness. Other parameter values are taken from Table 1. Interestingly, the system with stickiness becomes more stable for increasing a, while this dependence for the non-sticky system is non-monotone since $|\lambda_e|$ has a minimum at $a \approx 0.8$.

The range of output gap equilibrium values is proportional to the ratio of parameters b_1 and b_2 according to (15). Fig. 13 presents the dependence of $|\lambda_{i,e}|$ on these parameters.

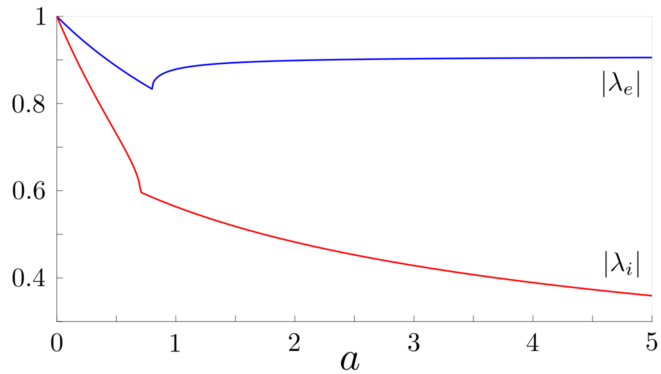
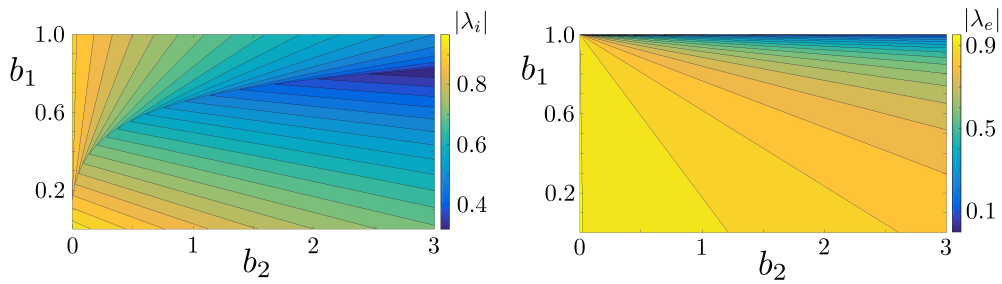


Figure 12: Variation of $|\lambda_i|$ and $|\lambda_e|$ with a. Other parameters are taken from Table 1.

The sticky system is more stable than its non-sticky counterpart for $b_1 < 0.9$, but becomes less stable than the non-sticky system as b_1 approaches 1 (in the latter case, the future inflation rate is defined predominantly by expectations). The dependence of $|\lambda_{i,e}|$ on b_2 and the dependence of $|\lambda_e|$ on b_1 is monotone (stronger stability for larger $b_{1,2}$), while the dependence of $|\lambda_i|$ on b_2 is non-monotone. The strongest stability is achieved by the sticky system for some intermediate value of b_1 between 0 and 1.



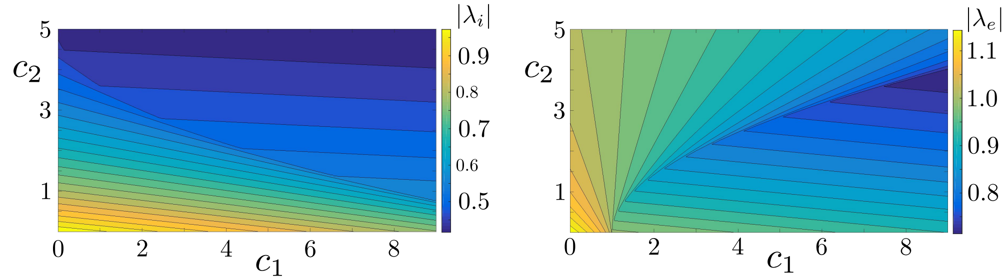
(a) (b)

Figure 13: Dependence of (a) $|\lambda_i|$ and (b) $|\lambda_e|$ on b_1 and b_2. Other parameters are taken from Table 1.

Parameters c_1 and c_2 control the range of inflation rate equilibrium values according to (15). This range contracts when c_1 increases (for $c_1 > 1$) and expands when c_2 increases. Fig. 14 shows that the sticky system is generally more stable than the non-sticky one. Both systems become more stable with increasing c_1 (stronger inflation targeting in Taylor's rule), see Figs. 14(a, b) and 15(a, b). The dependence of $|\lambda_i|$ on c_2 demonstrates some slight non-monotonicity for large c_2 values, see Figure 15(b). The non-monotonicity of $|\lambda_i|$ with c_2 is much more pronounced with the minimum achieved for a certain value of c_2 depending on c_1, see Figs. 14(b) and 15(b). This minimum corresponds to the strongest stability and, in this sense, optimizes the Central Bank policy. In Fig. 14(b), the strongest stability is achieved on the 'parabolic' line.

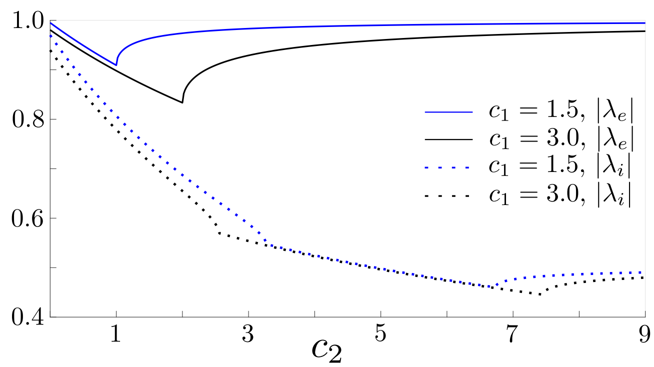
D. *Inversion of the PI operator*

In this section, we consider the inversion of the PI operator, which is necessary to transform the implicit system (5), (6) coupled with relation (32) into the explicit form (35). Here

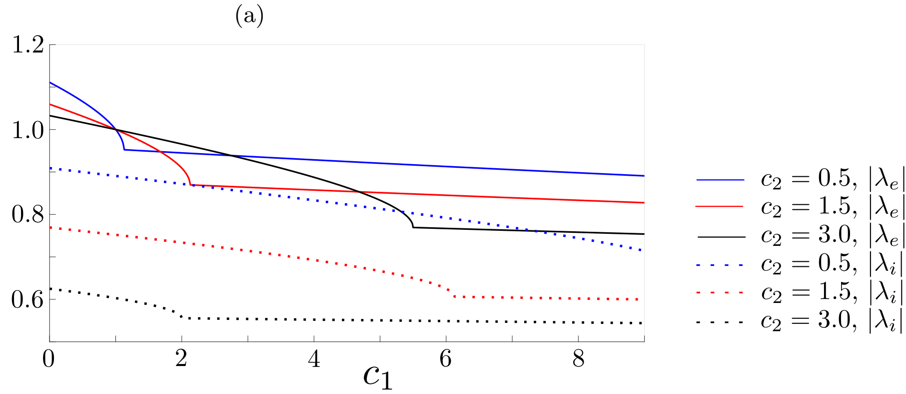


(a) (b)

Figure 14: Dependence of (a) $|\lambda_i|$ and (b) $|\lambda_e|$ on c_1 and c_2. Other parameters are taken from Table 1.



(a)



(b)

Figure 15: Cross-sections of the plots shown in Fig. 14 (a) for various c_2 values and (b) for various c_1 values.

we use the term 'PI operator' for an input-output relationship of the form

$$f_t = \alpha x_t + \sum_{i=1}^{n} \mu_i \mathcal{S}_{\rho_i}[x_t], \tag{27}$$

where the weights μ_i are allowed to have any sign, $\alpha \geq 0$, and $\rho_1 < \rho_2 < \cdots < \rho_n$. Such an operator is completely defined by the so-called *Primary Response* (PR) *function* $\phi(x)$, which describes the output in response to a monotonically increasing input. Here, this is a

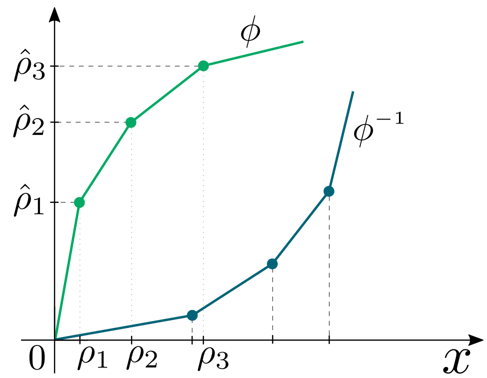


Figure 16: PR function ϕ of PI operator (27) and PR function ϕ^{-1} of its inverse PI operator (28).

piecewise linear continuous function satisfying $\phi(0) = 0$ with the slopes defined by

$$\phi'(x) = \begin{cases} \alpha + \mu_n + \cdots + \mu_2 + \mu_1, & 0 < x < \rho_1, \\ \alpha + \mu_n + \cdots + \mu_2, & \rho_1 < x < \rho_2, \\ \vdots & \\ \alpha + \mu_n, & \rho_{n-1} < x < \rho_n, \\ \alpha, & x > \rho_n, \end{cases}$$

see Fig. 16. As shown in [33], if the slopes of ϕ are all positive, then the PI operator (27) is invertible, and the inverse relationship is also a PI operator:

$$x_t = \hat{\alpha} f_t + \sum_{i=1}^{n} \hat{\mu}_i \mathcal{S}_{\hat{\rho}_i}[f_t]. \tag{28}$$

Further, the PR function of operator (28) is the inverse of the PR function ϕ of operator (27). This allows one to express the weights $\hat{\alpha}, \hat{\mu}_i$ and the thresholds $\hat{\rho}_i$ explicitly in terms of the weights α, μ_i and the thresholds ρ_i. In particular, the equation $\alpha x_t + s_t = f_t$ with $s_t = \mathcal{S}_\rho[x_t]$ (see (24)) can be inverted as

$$x_t = \frac{1}{\alpha} f_t - \frac{1}{\alpha(1+\alpha)} \mathcal{S}_{(1+\alpha)\rho}[f_t],$$

and this implies $s_t = \frac{1}{1+\alpha} \mathcal{S}_{(1+\alpha)\rho}[f_t]$, which is equivalent to (10) (cf. Appendix A).

E. Sticky Taylor rule

In order to convert system (19), (20) to the explicit form, we replace the variable y_t with the variable $g_t = c_1 x_t + c_2 y_t$ and obtain

$$g_t = (c_1 + a c_2) x_t + g_{t-1} - c_1 x_{t-1} - a c_2 \mathcal{P}_\sigma[g_t] + c_2 \epsilon_t, \tag{29}$$

$$x_t = \frac{c_2(1-b_1)}{b_2 c_1 + c_2(1-b_1)} x_{t-1} + \frac{b_2}{b_2 c_1 + c_2(1-b_1)} g_t + \frac{c_2(1-b_1)}{b_2 c_1 + c_2(1-b_1)} \eta_t. \tag{30}$$

Further, substituting (30) into (29) gives

$$\alpha g_t + \kappa \mathcal{P}_\sigma[g_t] = f_t \tag{31}$$

with

$$\alpha = \frac{c_2(1-b_1-ab_2)}{b_2c_1+c_2(1-b_1)}, \qquad \kappa = ac_2,$$

$$f_t = g_{t-1} - c_1x_{t-1} + \frac{c_2(1-b_1)(c_1+ac_2)}{b_2c_1+c_2(1-b_1)}(x_{t-1}+\eta_t) + c_2\epsilon_t.$$

Using that $\alpha > 0$ due to (21), we can invert (31) as in Appendix D to obtain

$$g_t = \frac{1}{\alpha}\left(f_t - \frac{\kappa}{\alpha+\kappa}\mathcal{P}_{\alpha\sigma}[f_t]\right).$$

This equation together with (30) defines the explicit system for (19), (20). The linearization $z_t = Bz_{t-1}$ of this system at any equilibrium point with $s_* \in (-\sigma, \sigma)$ has the matrix

$$B = \frac{1}{1-b_1-ab_2}\begin{pmatrix} 1-b_1 & a(1-b_1) \\ b_2 & 1-b_1 \end{pmatrix}.$$

Since

$$\det B = \frac{1-b_1}{1-b_1-ab_2} > 1,$$

all these equilibrium states are unstable.

F. A multi-agent model

Model (7) can be easily extended to account for differing types of agent with different inflation rate expectation thresholds. To this end, we replace the simple relationship (4) between p_t and x_t with the equation

$$p_t = \sum_{i=1}^{n} \mu_i \mathcal{P}_{\rho_i}[x_t] = x_t - \sum_{i=1}^{n} \mu_i \mathcal{S}_{\rho_i}[x_t] \tag{32}$$

with

$$\sum_{i=1}^{n} \mu_i = 1. \tag{33}$$

Here the play operator $\mathcal{P}_{\rho_i}$ models the expectation of inflation by the i-th agent; p_t is the aggregate expectation of inflation; $\mu_i > 0$ is a weight measuring the contribution of agent's expectation of inflation to the aggregate quantity; and, ρ_i is an individual threshold characterizing the behavior of the i-th agent. Relation (32) is equivalent to the formula

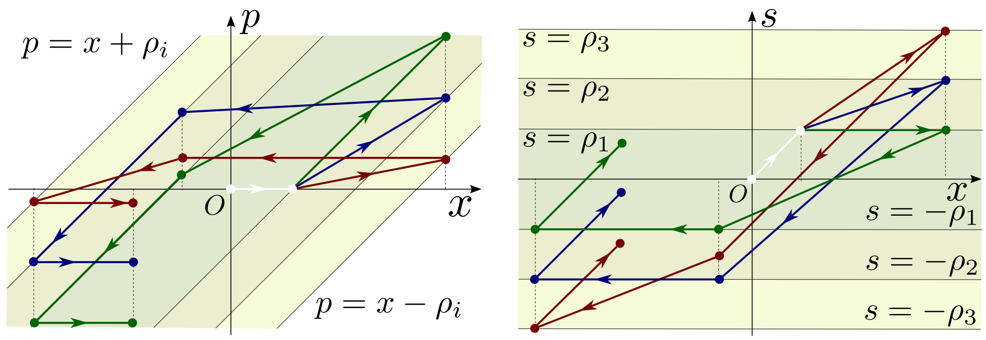
$$s_t = \mathcal{I}[x_t] := \sum_{i=1}^{n} \mu_i \mathcal{S}_{\rho_i}[x_t], \tag{34}$$

which is a (discrete) Prandtl-Ishlinskii (PI) operator with thresholds ρ_i and weights μ_i [34, 35, 36], where $s_t = x_t - p_t$.

The implicit system (5), (6), (32) with multiple agents can be converted into an explicit form using the same technique as we used for the system with one play operator. Again this involves the inversion of the PI operator. The explicit system

$$z_t = Az_{t-1} + \hat{\mathcal{I}}[c \cdot z_{t-1} + \hat{\xi}_t]\, d + N\xi_t, \tag{35}$$

which is similar to its counterpart (7), includes a PI operator with rescaled thresholds $\hat{\rho}_i$ and weights $\hat{\mu}_i$, see Appendix D for details; ξ_t, $\hat{\xi}_t$ denote the noise terms.



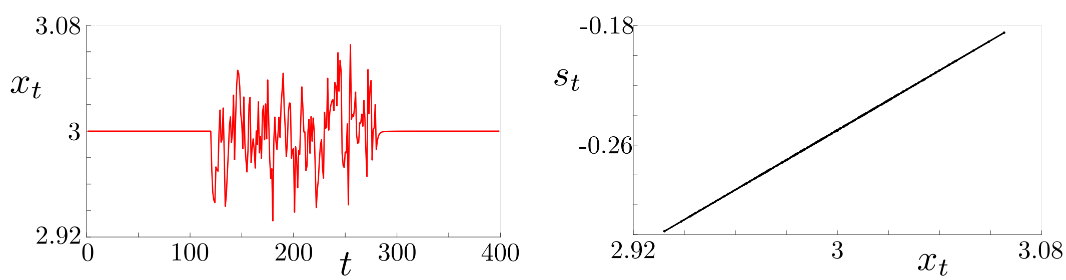
(a) (b)

Figure 17: Different expectations of agents based on three thresholds $\rho_1 < \rho_2 < \rho_3$ of (a) play and (b) stop operators with a single input x_t.

The stability properties of the equilibrium states of system (35) with multiple agents are similar to the stability properties considered above in Section 2.4. In particular, if we consider the system without external noise for $c_1 > 1$, then the set of equilibrium states is globally stable, and every trajectory converges to an equilibrium state.

In the simulations of this section, we classify economic agents into three categories, strongly, moderately, and weakly sensitive to inflation rate variations (hence $n = 3$), by assigning thresholds $\rho_1 < \rho_2 < \rho_3$, respectively, to these groups, see Fig. 17. Further, the contribution of each group to the aggregate expectation of inflation carries equal weight, $\mu_i = 1/3$.

Overall, numerical results obtained for model (5), (6), (32) with three agents are qualitatively similar to the results described above for the model with one agent, see Figs. 18 – 25, which are counterparts of Figs. 4 – 10, respectively.



(a) (b)

Figure 18: Trajectory of the system with 3 agents near an equilibrium state when none of the agents achieves an extreme perception gap (cf. Figure 4(a, d)). Here $c_1 > 1$. (a) Time trace of inflation. (b) Inflation versus expectation of inflation by any of the agents.

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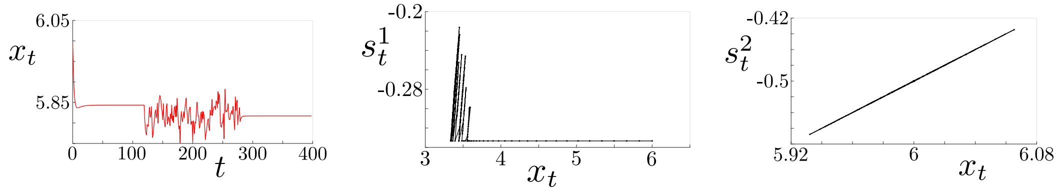
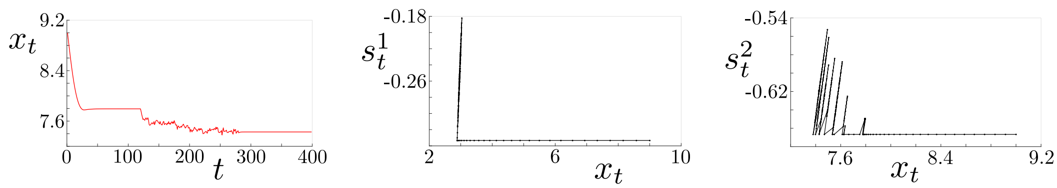


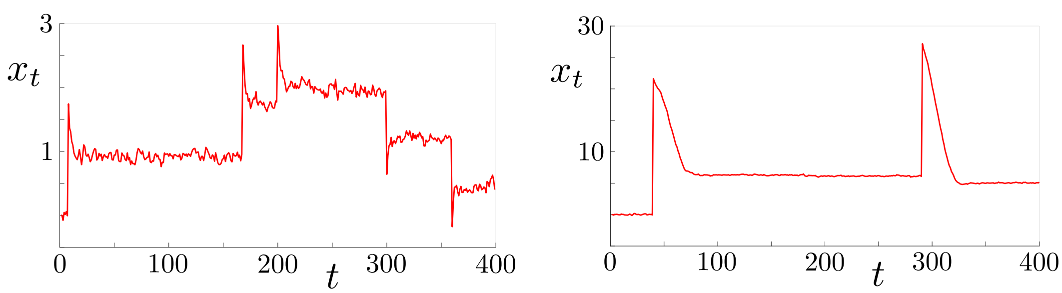
(a) (b) (c)

Figure 19: Trajectory of the system with 3 agents when the most sensitive agent reaches an extreme perception gap but the two less sensitive agents do not (cf. Figure 4(b, e)). The parameter c_1 satisfies $c_1 > 1$. (a) Time trace of inflation. A change of the equilibrium state occurs. (b) Inflation versus expectation of inflation by the most sensitive agent. (c) Inflation versus expectation of inflation by each of the two less sensitive agents.



(a) (b) (c)

Figure 20: Trajectory of the system with 3 agents with the most sensitive agent and the moderately sensitive agent having an extreme perception gap at the initial (equilibrium) point (cf. Fig. 4(c, d)). The parameter c_1 satisfies $c_1 > 1$. (a) Time trace of inflation. (b) Inflation versus expectation of inflation for the moderately sensitive agent. (c) Inflation versus expectation of inflation for the most sensitive agent. The least sensitive agent shows the behavior as in Fig. 19(c).



(a) (b)

Figure 21: Changes of the equilibrium state in the model with 3 agents due to shocks (cf. Figures 5, 6). (a) Small shocks. (b) Relatively large shocks.

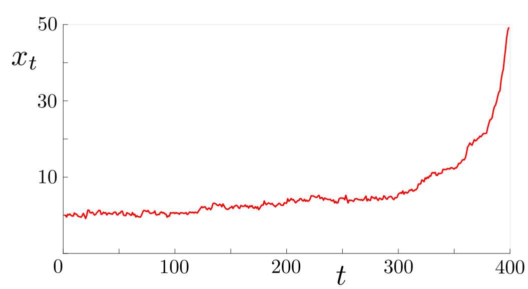
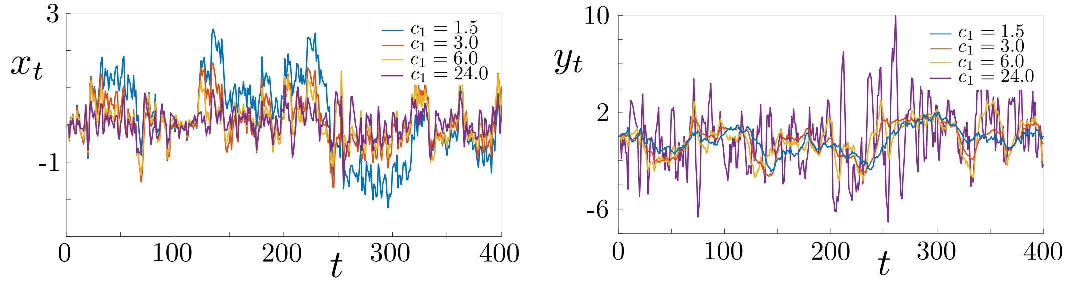
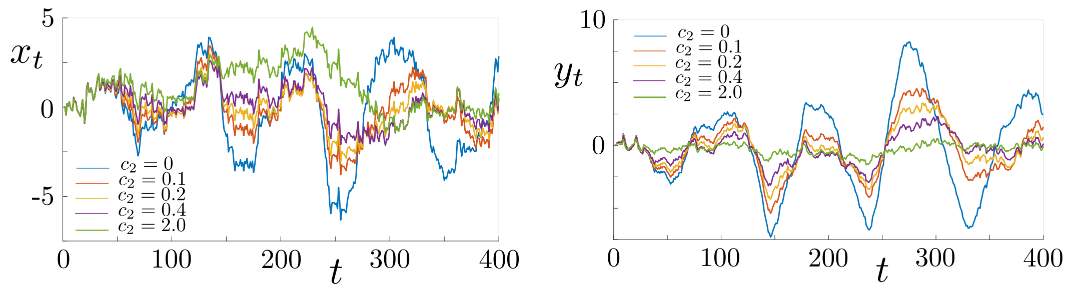


Figure 22: The run-away inflation scenario in the model with 3 agents in the case $c_1 < 1$ (cf. Fig. 7).



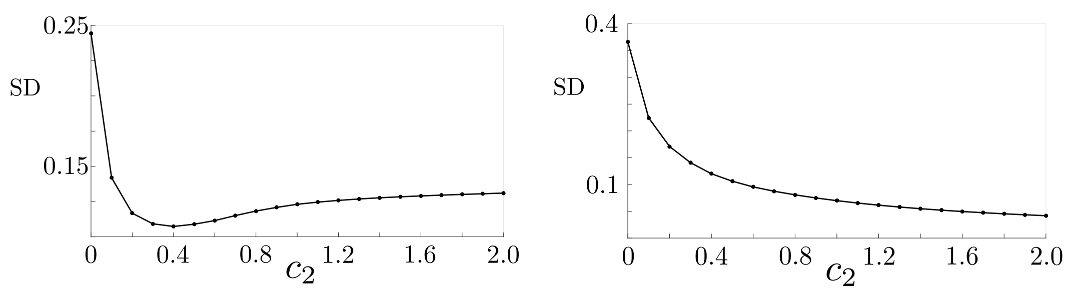
(a) (b)

Figure 23: Trade-off between the inflation and output gap volatility in the model with 3 agents as the inflation targeting parameter c_1 in the Taylor rule is varied (cf. Fig. 8). (a) Trajectories of x_t. (b) Trajectories of y_t.



(a) (b)

Figure 24: Trade-off between the inflation rate and output gap volatility in the model with 3 agents as the output gap targeting parameter c_2 in the Taylor rule is varied (cf. Fig. 9). (a) Trajectories of x_t. (b) Trajectories of y_t.



(a) (b)

Figure 25: Measure of the effect of c_2 on volatility of (a) inflation rate, x_t and (b) output gap, y_t with standard deviation (SD) (cf. Fig. 10).

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