FRUSTRATION MINIMIZATION, HYSTERESIS, AND THE EL FAROL PROBLEM

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ABSTRACT : The parable, due to Arthur, of the El Farol bar provides an account of the aggregate dynamics resulting from individuals using heterogeneous inductive predictors when deciding whether or not to attend the bar. In the Arthur formulation individuals make their bar attendance decisions on the basis of observing how crowded the bar has been in the preceding weeks. We modify this approach to take account of the times an individual experienced the enjoyment of an uncrowded bar and the regret at either having been at a too crowded bar, or at not having come when the bar was uncrowded. In our formulation the attendance decision is driven by the dominant component of enjoyment or regret. We allow for hysteresis thresholds and our simulations show that this modification to our model leads to an increase of the periodicity of aggregate bar attendance.

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1 Introduction

In this paper we introduce a simple conception of human motivation and investigate its consequences for the original formulation of the El Farol bar problem of W. Brian Arthur (1994). The decision facing the individual is whether or not to attend the El Farol bar in view of the possibility that it might be too crowded. In the Arthur representation, individuals observe the aggregate dynamics of bar attendance and choose from a set of predictors the ones that have most accurately predicted the actual recent bar attendance. The "active" or successful predictors are selected by a process of induction (see section 2 for more details).

This approach contrasts with the deductive approach to the "rationality" of decision-taking traditionally used in economics, such as in the expected utility theory: see Chris Starmer (2000) and Daniel Kahneman (2003) for critical survey assessments. In the traditional approach, "rationality" is defined in the instrumentalist sense (Robert Sugden, 1991) of consistency with a set of simple axioms such as those articulated by Leonard J. Savage (1954). Many violations of such axioms have been observed (see Isabelle Brocas and Juan D. Carrillo, 2003). Of the alternative explanations of decision-taking, prospect theory (Kahneman and Amos Tversky, 1979), regret theory (Graham Loomes and Sugden, 1982) and cognitive dissonance (George A. Akerlof and William T. Dickens, 1982) are amongst the most prominent.

The innovation in the present paper is to introduce some layers of psychology into the inductive learning involved in the Arthur (1994) formulation: see George Loewenstein (1992). Our procedure is to introduce the insights of the regret theory of David Bell (1982), Peter C. Fishburn (1982), and Loomes and Sugden (1982) into the decision whether or not to attend the El Farol bar. This is done by taking into account the times the individual experienced the enjoyment of a non-crowded bar, the regret or disappointment at having attended the bar when it was too crowded, and the lost opportunities associated with not having attended when the bar was not crowded. The cognitive dissonance of Akerlof and Dickens (1982) is introduced by allowing the individuals to have hysteresis thresholds with respect to enjoyment, disappointment, and lost opportunities. Thus attendance strategies can be retained even when the bar attendance dynamics that led to the strategies have been removed. While taking regret and cognitive dissonance into account, our analysis attempts to retain a strong flavour of the induction present in the original Arthur (1994) formulation. The behaviour described is rational only in the "bounded" sense of Herbert A. Simon (1955). We contend that rationality in the El Farol conditions of incomplete information requires not only inductive learning but also knowing one's psychological requirements, that is, knowing oneself (see Roland

Bénabou and Jean Tirole, 2003). We feel that Socrates would have agreed.

The resulting model is simple to state and relatively easy to simulate on a computer. Its analysis, due to the discontinuous strategy-switching which is a necessary consequence of our assumptions, is non-trivial. There are a myriad of possible modifications and elaborations of the basic setup that can be further explored.

The structure of the paper is as follows. In Section 2 we briefly review Arthur's formulation and approach to the El Farol problem. This is not so much a "problem" as a benchmark example which can be used to see how different approaches to the decision-making of individuals are reflected in the aggregate dynamics (occupation dynamics of the bar in the El Farol case). In Section 3 we put forward our approach to human motivation. In Section 4 we express cognitive dissonance in terms of our key variables and show that this gives rise to hysteretic behaviour at the level of the individual. Finally, in Section 5 we present results of some simulations of the model that indicate a huge richness of dynamics arising from our very simple assumptions; this, as work on even low-dimensional dynamical systems (e.g., Anatole Katok and Boris Hasselblatt, 1999) shows, is not unexpected.

2 The Arthur Approach

In his classic paper Arthur (1994) considers the following situation: there is only one bar in the town, the El Farol bar, and every citizen over licensing age, of which there are 100, has each week to make a decision whether or not to go to the bar on Thursday night, when they play Irish music. It is known that if the number of occupants in the bar on a particular night is larger than some critical number, say 60, no-one will have a good time (too crowded, service will be slow, etc.). The question is to understand how reasonable assumptions on the decision-making procedures used by the individuals are expressed in the occupancy dynamics of the bar.

Arthur's approach to the problem is as follows. He assumes that there is a set of n predictors $P = \{p_1, \ \ldots, \ p_n\}$, of which the j-th individual ($j = 1, \ldots, 100$) is "issued" with a subset P_j of cardinality $m < n$. Each of these predictors uses information on past occupancy to predict the occupancy next Thursday. Each individual picks, at each decision time (say, on Thursday afternoon), the prediction of the predictor that had done best the previous Thursday afternoon in predicting the occupancy on that Thursday night, and acts accordingly: if the prediction is that the number of people will be larger than 60, the individual in question will not go, and so on. Now, even though not taking into account the overall record of predictors and judging by last week's performance only, such a choice of a predictor on which to base a decision can be defended on the grounds that the individual is obliged to learn inductively in the absence of any deductive means of figuring out the aggregate bar attendance.

Before we suggest an arguably more realistic alternative, we would like to introduce a distinction between **data** and the **construction** placed on the data. Henceforth by data we will just mean the facts relevant to the situation and by the construction, data filtered by the individual and put into a form useful for decision-making. Thus, data in Arthur's approach consists of the history of the occupancy of the bar in previous weeks. For simplicity, we will assume infinite memory of the individuals; this is of course not realistic. Finite memory clutters the notation, but can be incorporated. If N_k is number of people in the bar in week k, after week i the data consists of the vector

$$\boldsymbol{N}_i = \{N_1 \, \ldots \, N_i\}. \tag{1}$$

(Thus, for any given i, a predictor $p \in P$ takes $\boldsymbol{N}_i$ as its input and predicts N_{i+1}.)

The construction for individual j after week i therefore consists of a number l defining the predictor $p_l \in P_j$ that best predicted the occupancy at the previous Thursday, i.e. the number l that minimizes $|p_l(\boldsymbol{N}_{i-1}) - N_i|$ and the number $p_l(\boldsymbol{N}_i)$.

Note that in this case data is shared, while the construction is private, since the sets P_j are different from individual to individual, and in fact, the sum total of individual differences is contained in the sets P_j.

3 An Alternative Approach to Decision Making

We would like to suggest a different picture of decision making. We introduce a layer of psychology and hereby attempt to populate our model with homo sapiens (see Richard H. Thaler, 2000). In our view, people are subject to multiple tensions and frustrations, which provide the motivation for action. One could organize the different tensions into one composite function, and try to minimize it, but we feel that the consequent mathematical simplicity has not much to commend itself. Such tensions define the psychological state of an individual as a vector only in a very high-dimensional space. A person's actions, in our view, are mainly of the "fire-extinguishing" variety: an individual will always attend to the dominant component of the vector, that is, will try to relieve the most prominent tension. Anyone who has sat on a needle while having tooth-ache will intuitively see the reason for this assumption.

We will now spell out the consequences of such assumptions for the El Farol situation. We are careful at each stage to distinguish between data and the psychological construction placed on it.

3.1 Data

Clearly, Arthur's model overlooks an obvious source of data: in addition to $\boldsymbol{N}_i$, which is data shared between all the individuals, each individual j also has access to her own record of bar attendance. We define the variable s_j^i

to be 1 if individual j was in the bar in week i and 0 otherwise. Thus, individual j also knows $\boldsymbol{s}_j^i = \{s_j^1, \ldots, s_j^i\}$ and can use that when formulating a bar attendance strategy.

3.2 Construction

Consider the following numbers:

$$D_j^i = \sharp\{l \in \{1, \ldots, i\} |\ N_l \geq 60,\ s_j^l = 1\}, \tag{2}$$

$$L_j^i = \sharp\{l \in \{1\ \ldots, i\} |\ N_l < 60,\ s_j^l = 0\}, \tag{3}$$

$$E_j^i = \sharp\{l \in \{1\ \ldots, i\} |\ N_l < 60,\ s_j^l = 1\}. \tag{4}$$

Here $\sharp$ denotes the cardinality of a set. D_j^i counts the times up to and including the i-th week, when the individual did not have a good time in the bar (disappointment), L_j^i is the count of lost opportunities to have a good time, and E_j^i is the number of times when the individual enjoyed herself in the bar.

So far no account has been taken of human variability. Clearly, people differ in the way past experience impinges on their self-perception. One way to encode this is to give different weights to occurrences depending on their distance in time from the present. That is a possibility, but we shall pursue a different approach, by noting that some people attach little weight to lost opportunities while others attach more weight to disappointments than to good times. Hence we associate each of the numbers D_j^i, L_j^i and E_j^i with positive weights d_j, l_j, e_j (see Starmer, 2000). The construction used to make a decision is encoded in the **state of the individual** j,

$$\boldsymbol{S}_j^i = (D_j^i d_j,\ L_j^i l_j,\ E_j^i e_j). \tag{5}$$

3.3 Decision Making

One way to define "quasi-rational" (in the sense of Thaler, 2000) behaviour for individual j would be to have her maximize $E_j^{i+1} e_j$, and simultaneously minimize $D_j^{i+1} d_j$ and $L_j^{i+1} l_j$; or, alternatively, to maximize an additive expression such as

$$E_j^{i+1} e_j - L_j^{i+1} l_j - D_j^{i+1} d_j, \tag{6}$$

given $\boldsymbol{S}_j^i$, that is, to make a choice of s_j^{i+1}. Were we to follow this approach, we would have mainly to explain the j-th individual's (quasi-rational) prediction for the number of visitors in week $i+1$, $\overline{N}_j^{i+1}$. Then, with slight modifications, this approach would collapse to that of Arthur, notwithstanding the layer of psychology that we have introduced.

However, if the "fire-extinguishing" approach suggested above is to be followed, it makes sense to define

$$C_j^i = \max\{D_j^i d_j,\ L_j^i l_j,\ E_j^i e_j\}. \tag{7}$$

Our representation of human motivation is then to say that the bar attendance decision is determined simply by C_j^i: if $C_j^i = E_j^i e_j$ or $C_j^i = L_j^i l_j$, then $s_j^{i+1} = 1$, and if $C_j^i = D_j^i d_j$, $s_j^{i+1} = 0$. Of course one could introduce probabilities to take into account inertia, but even without that the resulting well-defined (non-smooth) dynamical system seems interesting enough. Thus our model is iterated as follows: at time i compute C_j^i for all the individuals and update s_j^{i+1}. The parameters could in principle be calibrated by experimental economics techniques.

We are not saying that the individuals are not anticipatory systems in the sense of Robert Rosen (1985), in that they do not carry within themselves a predictive model of the environment. We just claim that our model only takes account of psychological constructions placed on different predictions. If what makes the individual suffer is the thought of lost opportunities, she will try to alleviate this suffering by making sure the weight of lost opportunities does not become heavier, even at the expense of disappointment from visiting an overcrowded bar.

It is not clear that, in relation to challenges occurring over a relatively short time-span, human behaviour should be expected to have been shaped by evolutionary forces (see the discussion of James H. Fetzer, 1990). Attending to a pressing psychological need at the expense of a less pressing one can have survival value, though obviously does not encompass the whole gamut of behavioural strategies that would be "rational" in an evolutionary sense. It is possible to find parameter values for which the average bar occupancy is around 60%, in which case the behaviour of the cohort as a whole can be deemed "rational" (since on the average the bar is neither overcrowded nor underused). We do not restrict ourselves *a priori* to such parameter regimes, since our aim is to model actual, and not "desirable" behaviour.

Clearly, as more information becomes available, it can be incorporated in the decision-making process. Thus, if a person is driven by disappointment and at the same time knows for certain that the bar will be empty next Thursday night, there is no reason why she should not go. In the simple setup we are considering, just as in life, such infallible oracles do not exist.

4 Hysteresis

The possibility of hysteretic behaviour is mentioned by Arthur but without elaboration. For a discussion of hysteresis in economics see Cross (1993), Laura Piscitelli (1998), or Grinfeld, Piscitelli and Cross (2000) for hysteresis set in a probabilistic context. In the present context, hysteresis involves the retention of a strategy once the stimulus that led to the adoption of the strategy has been removed. This is recognizably human: the expression in the economic arena of cognitive dissonance and habit formation.

We now describe one possible method of introducing hysteretic behaviour in terms of our variables: an individual is hysteretic with respect, say, to disappointment if

$$C_j^i = D_j^i d_j \quad \text{though} \quad D_j^i d_j \neq \max\{D_j^i d_j,\, L_j^i l_j,\, E_j^i e_j\} \tag{8}$$

$$\text{whenever} \quad D_j^{i-1} d_j = \max\{D_j^{i-1} d_j,\, L_j^{i-1} l_j,\, E_j^{i-1} e_j\}, \tag{9}$$

$$\text{and} \quad \max\{D_j^i d_j,\, L_j^i l_j,\, E_j^i e_j\} - D_j^{i-1} d_j < \epsilon_j, \tag{10}$$

where ϵ_j is the hysteresis threshold, the strength of j-th individual's cognitive dissonance (with respect to disappointment in this case).

It should be clear that if people are hysteretic with respect to enjoyment or lost opportunities (i.e. envy) occupancy levels at the bar will be higher than if there is no hysteresis. This indicates that a successful bar management strategy would try to induce and keep high hysteresis levels with respect to these two tension sources.

5 Numerics

We now present some numerical simulations using the above model, both with and without hysteresis. The range of dynamical behaviour that can be observed is very large depending upon the parameters entered into the model. Of particular interest is the observation that the overall occupancy of the bar from week-to-week can display approximate periodicity even though a significant percentage of the participants do not show this periodicity in their individual behaviour. Chaotic (no discernible periodicity) and intermittent (switching between two apparently stable but distinct modes of behaviour at seemingly random times) parameter regimes can also be observed. We also examine the effects of introducing hysteresis on the behaviour of individuals and on the overall bar occupancy rate.

In what follows we simulate the behaviour of 100 individuals and set the optimum occupancy rate of the bar at 60. This leaves us with 300 parameters to be chosen, namely the weights $d_j, l_j, e_j,\ 1 \leq j \leq 100$. We define these by drawing each d_j from a uniform probability distribution defined on a certain interval $[a_d, b_d]$. The same is done for each l_j and e_j using intervals $[a_l, b_l]$ and $[a_e, b_e]$ respectively. Thus, in a statistical sense, the model is reduced to one with only 6 parameters. While this method of assigning weights is certainly simplistic, it is sufficient to display a wide range of system dynamics and also allows for a controlled exploration of the parameter space by, for example, changing the average value of d_j and observing the effects.

The parameters used for the first simulations are $[a_d, b_d] = [3, 6], [a_l, b_l] = [1, 3]$ and $[a_e, b_e] = [3, 4]$. At the start of the simulation the values of D_j, L_j and E_j are randomly assigned to be either 1,2 or 3 to define an initial attendance history for the j^{th} individual. The model was then iterated for 100 weeks.

The first row of Figure 1 shows two plots – the left plot is a plot of the attendance for the last 30 weeks displaying the asymptotic or long-time dynamics after any initial transients have decayed away. The right plot shows the attendance record of 20 randomly selected individuals over the same 30 week period. Circles denote attendance while the absence of a circle denotes abs(tin)ence for that particular week.

After 100 weeks, hysteresis was introduced into the model which was then iterated for a further 100 weeks. This hysteresis is with respect to disappointment, loss and enjoyment, that is to say

$$C_j^i = \max\{D_j^{i-1} d_j,\, L_j^{i-1} l_j,\, E_j^{i-1} e_j\} \tag{11}$$

$$\text{if} \quad \max\{D_j^i d_j,\, L_j^i l_j,\, E_j^i e_j\} - \max\{D_j^{i-1} d_j,\, L_j^{i-1} l_j,\, E_j^{i-1} e_j\} < \epsilon_j. \tag{12}$$

For simplicity $\epsilon_j = 2$ for all j. The lower plots of Figure 1 were created in exactly the same way as the upper ones but show the last 30 weeks of the second 100-week period. It should be noted that the weights d_j, l_j and e_j are precisely the same for both sets of plots and the 20 randomly selected individuals are also the same. This permits a much more direct comparison of the effects of cognitive dissonance/hysteresis.

The most striking effect of including hysteresis is an increase in the approximate period of the aggregate behaviour. This was observed in a large majority of the simulations that were run, and is consistent over a very large parameter range.Also, as noted above, the behaviour of individuals is much less predictable than the overall bar-attendance. As can be seen from Figure 1 a wide variety of bar-attending behaviour naturally emerges, ranging from people who are present almost every week to those who turn

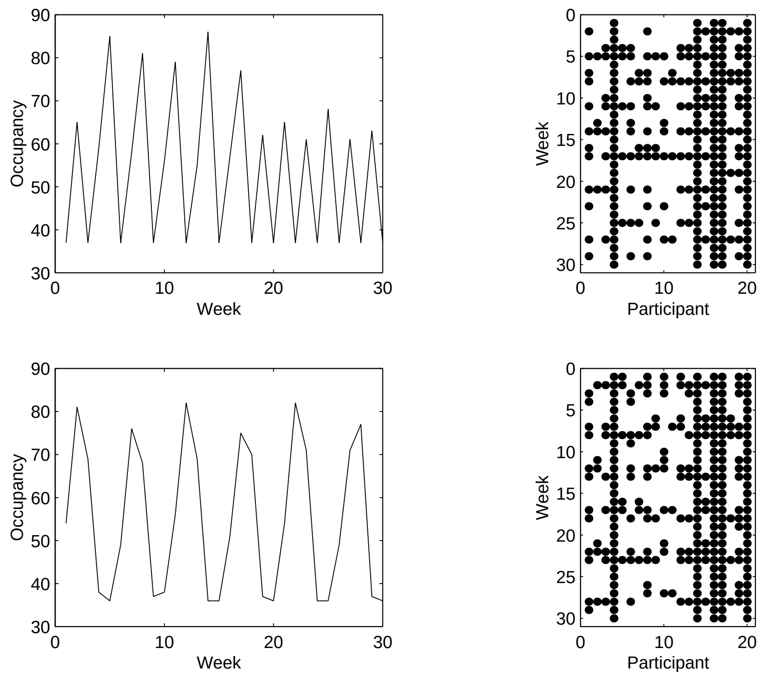

Figure 1: The top row shows the behaviour of the system without hysteresis while the lower row shows the effects of including hysteresis.

up very rarely.

Figure 2 is produced in exactly the same way as Figure 1 but with new parameters $[a_d, b_d] = [6, 7]$, $[a_l, b_l] = [2, 3]$ and $[a_e, b_e] = [3, 10]$. The results without hysteresis are similar to those in Figure 1 with an approximate periodic component of length 2-3 weeks. The addition of hysteresis once again appears to increase the length of any approximate periodicity that is present. However, the resulting aggregate attendance plot differs significantly from its counterpart in Figure 1 and forcefully demonstrates that even very simple frustration minimization strategies at the individual level can result in highly complex and unpredictable group dynamics.

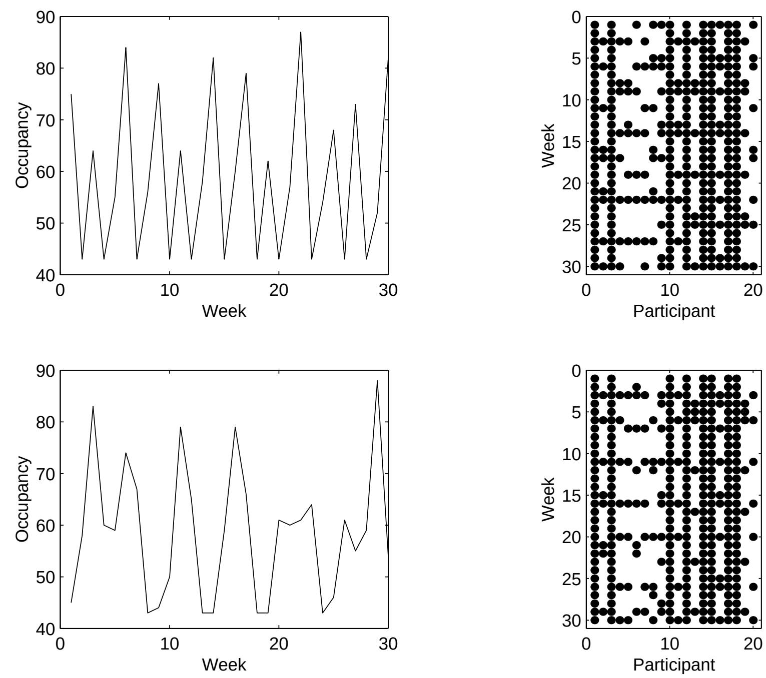

Figure 2: As Figure 1 but with data generated using different weighting parameters (see text).

6 Concluding Remarks

The noticeable degree of periodicity present in the numerical simulations of the previous section – perhaps higher than would be expected in reality – can be explained by the fact that the El Farol bar problem is extremely simplified. Following Arthur, we have made no distinction between there being 61 patrons in the bar and 100. Clearly, in realistic situations the quality of the service, amount of noise, etc. depend "continuously" on the number of people, and one would expect that having been present in a bar with another 99 punters, one would give it a wide berth for a long time. It is reasonably obvious how to incorporate such dependencies into our construction-creation mechanisms by making d_j time-dependent through dependence on N_i, and so on. Other shortcomings, such as the absence of external random influences and the infinite memory of the participants can also be incorporated.

We would, however, argue that by endowing our El Farol punters with a capacity for disappointment and regret at lost opportunities, and by allowing for cognitive dissonance and stickiness in habits, the bar becomes a more recognizable place. It is also plausible to allow the punters to remember whether or not they attended the bar (assuming they were not too drunk for that).

We finally remark that Arthur's model can be regarded as a parable that applies to a wider class of problems. It could, for example, be applied to firms making market entry-exit decisions, in which case the different predictors stand for the opinions expressed around the different executive board tables. Our notions of disappointment, lost opportunity, enjoyment and frustration minimization extend quite naturally to such situations.

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