

A mean-field model of investor behaviour

R Cross¹, M Grinfeld² and H Lamba³

¹Department of Economics, University of Strathclyde, Sir William Duncan Building, 130 Rottenrow Glasgow G4 0GE, Scotland, UK

²Department of Mathematics, University of Strathclyde, Livingstone Tower, 26 Richmond Street, Glasgow G1 1XH, Scotland, UK

³Department of Mathematical Sciences, George Mason University, MS 3F2, 4400 University Drive, Fairfax, VA 22030 USA

E-mail: `rod.cross@strath.ac.uk`, `caas05@maths.strath.ac.uk`, `hlamba@gmu.edu`

Abstract. In this note we investigate the ability of a mean-field model distilled from a heterogeneous agent model to simulate stylized facts of financial times series.

Keywords: heterogeneous agent models, mean-field approximations, economic modeling, market dynamics, herding.

1. Introduction

Since the early statistical studies of Bachelier [3], the degree of randomness of stock market prices has stimulated debates about the extent of stock market efficiency and about the underlying behaviour of investors, or their agents, that could explain the features of stock market price statistics. The efficient markets hypothesis (EMH) has, until recently, dominated the mainstream finance literature [9]. A typical Brownian motion formulation of the EMH has the stock log-price differences being identically and independently distributed (IID) across time, no matter what the time interval, and obeying a Gaussian distribution. The underlying behaviour is taken to be akin to a fair game in which investors exploit all the information relevant to the determination of stock prices so that the expected gain or loss is zero. The assumption is either that investors are homogeneous in the sense that they use the same information sets; or that informed investors quickly arbitrage away the gains to be had from the behaviour of uninformed investors [16]. The random arrival of new information that leads actual stock prices to differ from those expected by investors is translated into a Brownian motion in, and a Gaussian distribution of, stock returns.

A major problem with the EMH is that the returns on stocks, the log-price differences, are not well approximated by a Gaussian distribution [6]. The serial autocorrelation of returns is close to zero for returns defined over periods longer than about 15-30 minutes, but the Gaussian is not a good approximation for even the first moment of the returns distribution for difference periods of less than a month [10]. Two key departures from a Gaussian distribution are the fat tails and clustered volatility observed in actual returns distributions. The fat tails indicate that extreme returns occur more frequently than in a Gaussian world, with the tails displaying signs of scaling that are possibly consistent with power laws [10]. Clustered volatility indicates that the returns are not identically distributed, the variance changing over time. Volatility indices such as the US VIX chart the positive serial correlation observed in the second moments

or absolute values of returns and thus reveal periods of quiescence punctuated sporadically by episodes of high volatility.

The implications of these, and other [17], non-Gaussian features of stock returns have been the subject of intense debate. From proponents of the EMH there is a transactions cost defence: “While the stock market may not be a mathematically perfect random walk, it is important to distinguish statistical from economic significance—the statistical dependencies are not likely to permit investors to realise excess returns—anyone who pays transactions costs is unlikely to fashion a trading strategy based on the kinds of momentum found in these studies that will beat a buy-and-hold strategy” [15, p. 62]. The decline in trading costs arising from electronic trading [20] suggests that this defence has become less plausible.

It was perhaps the excess volatility observed in stock returns [18] that most stimulated interest in alternatives to the EMH account of investor behaviour. The point of departure in the behavioural finance (BF) approach was to point out the limits of arbitrage [19]. Stocks as a whole do not have close substitute portfolios, so if they are mispriced there is no riskless hedge for the arbitrageur. The risk that new information will be such that the stocks become even more mispriced exacerbates the problem, even assuming that investors can identify ‘fundamental’ values. In this world the behaviour of ‘noise’ investors is driven by changes in sentiment, as well as by the arrival of new information relevant to ‘fundamental’ stock values. If the changes in sentiment are correlated to some degree across investors, the ‘noise’ trades will not cancel each other out, and so there will be an impact on market prices. This framework raises problems with regard to ‘fundamentalist’ investors. Do they stick to ‘fundamentals’, or do they also take account of the impact on market prices of ‘noise’ investors? In this world, Keynes’ beauty contest metaphor [14] becomes not completely implausible.

An implication of the BF approach is that affective as well as cognitive processes can influence investor behaviour [5]. In a previous paper [7], we introduced affective responses in the form of ‘inaction’ and ‘herding’ thresholds to model investor psychology and market sentiment. The idea was to simulate an heterogeneous agent model (HAM) of the stock market in which investors are, in part, motivated to switch positions when certain psychological pressures become unbearable. These will be described in more detail in Section 2. The model was largely successful in replicating the main non-Gaussian features, but so have been other HAMs (see the review in [12]). The parsimonious representation of investor psychology in the HAM of [7] arguably has an edge over more complicated behavioural specifications in alternative HAMs. This still leaves unresolved the question of whether there is an underlying common mechanism for the generation of the non-Gaussian features of stock returns. As Farmer [10, p. 34] puts it, “the problem is not ‘how do we get realistic deviations from normality and IID behaviour?’ but rather, ‘how do we determine the necessary and sufficient conditions, and how do we identify which factors drive such effects in real markets?’ ”.

It is difficult to do much with HAMs by way of analysis due to their high dimensionality. Hence it makes sense to distill the salient features of HAMs, into a simpler model based on easily understandable principles and check whether the time series of price returns it generates would retain desirable properties. This process in general requires a significant dimension reduction, and can only be justified in a small number of situations.

One such situation is considered, among other places, in [7], where all the traders in a market are only coupled globally, through the price of the asset and through their (not necessarily complete) knowledge of market sentiment. In such a situation, one is allowed to pass to the mean field limit, as in Horst [13], where such a process is considered in detail; we will return to consider this model in detail in the next section.

In this paper we show that a distillation of psychological facts of [7] into statements of probabilities of transitions between market positions leads to a system of a stochastic difference equation coupled to a deterministic one, the statistics of which retain many of the observed

features of financial time series. The resulting framework can in principle be analysed by tools of random dynamical systems theory [2]; we leave this analysis for a later publication.

2. A mean-field model

A good starting point for our derivation is the model of Horst [13]. Even though this model cannot reproduce the key features of financial time series (see below), it is the most thorough and rigorous derivation of a mean-field model from a Markov chain one known to us; it is also very useful for comparison with our approach.

Consider a market for an asset with price p_t , at time t populated by (an infinity of) traders. Let the proportion of long at time t traders be denoted by m_t^+ , while that of short traders by m_t^- . Clearly, $m_t^+ + m_t^- = 1$, and we define an order parameter m_t to be $m_t := m_t^+ - m_t^-$. Thus m_t is a measure of the sentiment of the market; below we will refer to it as *the* sentiment.

Briefly, in [13] it is assumed that the price p_t is slaved to m_t ,

$$p_t = p_F + \alpha m_t, \quad (1)$$

where p_F is “the” fundamental price of the asset and α is some positive constant (a similar assumption is made in [1]), while the sentiment $m_t \in [-1, 1]$ evolves via an equation of the form

$$m_{t+1} = f(e_{t+1}, h_t). \quad (2)$$

Here $h_{t+1} = h_t + \eta_t$, $\eta_t \sim \mathcal{N}(0, \sigma_h)$, and corresponds to exogenous ‘rational’ market-changing information entering the system. The quantity $e_{t+1} = m_t + \epsilon_t$, with $\epsilon_t \sim \mathcal{N}(0, \sigma_\epsilon)$, represents uncertainty in the estimation of the market sentiment. The function f in (2) is of the same type as in (6) below. The resulting dynamics of m_t for reasonable choices of distributions of η_t and ϵ_t are roughly bimodal, which by (1) means that the price returns implied by this model cannot correspond to those observed in real markets.

We now derive a new mean-field model based partly upon (1) and (2) but also using ideas from previous HAMs, in particular, from the one of [7]. Firstly we note that in (1) the price p_t is a deterministic function of the sentiment m_t . This occurs because all the random effects, including the exogenous information h_t entering the marketplace, affect the sentiment directly in (2). Thus new information affects the sentiment which then affects the price in a deterministic way. However, we shall argue that new information entering the market affects the price directly. This is much closer to the spirit of rational market theory and the EMH where the concept of sentiment does not even exist. We leave a detailed justification of (3) to a subsequent publication, but note for now that it is plausible to suggest that as new information arrives, it is liable, for example, to result in changes of both bids and offer prices on limit orders (see [8] for more details on this terminology and a discussion of the underlying trade mechanism). Hence one can effect a change in the stock price virtually without any increase in transactions. Another way of saying this is that we shall model the exogenous noise as directly affecting the fundamental price p_F , rather than suppose p_F to be a constant. We also assume that the new updated price will also depend in a linear manner upon the change in sentiment over the last timestep (via the laws of supply and demand) resulting in the alternative pricing formula

$$p_{t+1} = p_t + \eta_t + \kappa \Delta m_t, \quad (3)$$

where $\Delta m_t := m_t - m_{t-1}$ and $\eta_t \sim \mathcal{N}(0, \sigma)$. This price updating model was employed in the HAM of [7] (note that if a geometric pricing model is being used then the price variable p_t is in fact the log-price but, for brevity, we shall simply refer to p_t as the price). Here the variance σ is chosen to reflect the magnitude of the external fluctuations occurring over one time period (whether that corresponds to days or months etc). The parameter κ is a measure of the liquidity

of the market and the extent to which pricing changes are affected by the arrival of external information as opposed to internally generated market dynamics. Finally we point out that setting $\kappa = 0$ leads to EMH dynamics of returns.

We now suggest an updating formula for the sentiment, similar in form to (2), but where the change in sentiment now depends upon the change in market price. In particular, the rules determining how each agent decides whether or not to switch are as follows. Firstly each agent has an ‘inaction’ threshold whereby when the price moves far enough away from the price at which she last traded, she will switch. This mimics many different (rational and irrational) market effects such as taking profits, cutting losses, or just an irrational need to do something. Intervals of this kind also arise naturally when one considers the effects of transaction/sunk costs that prevent agents switching arbitrarily often. The second threshold affecting each agent is a ‘herding’ threshold. Here agents can only tolerate being in the minority for so long, after which point they have to switch. This herding tendency is strongly suspected, via its incorporation into many different successful HAMs, as being a major contributor to the ‘boom-and-bust’ dynamics of financial markets. The reader is directed towards [7] for further details of the modeling and its justification. The numerical results presented there are interesting in that the herding tension is shown to be directly responsible for the fat-tails and excess kurtosis.

The master equation for the sentiment is

$$m_{t+1}^+ = m_t^+ P_{(+,+)} + m_t^- P_{(-,+)}, \quad (4)$$

where $P_{(+,+)}$, for example, is the probability of a long trader staying long.

Now, $m_t^+ = (m_t + 1)/2$ and $m_t^- = (1 - m_t)/2$, so we get

$$m_{t+1} = 2 \left(\frac{m_t + 1}{2} P_{(+,+)} + \frac{1 - m_t}{2} P_{(-,+)} \right) - 1. \quad (5)$$

Note that $1 = (m_t + 1)/2 + (1 - m_t)/2$. Now let us consider the transition probabilities. The range of the function $P_{(+,+)}$ should be $[0, 1]$. We also want this function to be monotone in m_t ; this is in agreement with the specification of the herding tendency of [7]. In addition, we want to allow a dependence on Δp_t . Similar considerations for $P_{(-,+)}$ lead to the following possible form of the two transition probabilities:

$$P_{(+,+)} = \frac{1}{2}(1 - \tanh \beta(-Jm_t + f_1(\Delta p_t))) \text{ and } P_{(-,+)} = \frac{1}{2}(1 - \tanh \beta(-Jm_t + f_2(\Delta p_t))),$$

where as in [4], J is a measure of the strength of social interactions. We shall leave f_1 and f_2 undetermined for now. Opening up the brackets in (5) and using the fact that the tanh function is odd, gives

$$m_{t+1} = \frac{m_t + 1}{2} \tanh \beta(Jm_t - f_1(\Delta p_t)) + \frac{1 - m_t}{2} \tanh \beta(Jm_t - f_2(\Delta p_t)) \quad (6)$$

Next we need to determine what the dependence of f_1 and f_2 on Δp_t should be in order to incorporate the inactivity tension described above. A simple and plausible choice is $f_1(x) = |x|$. This would mean that if there is a large jump in price, the probability $P_{(+,+)}$ is small: the traders will prefer to modify their market positions. Similarly, a good choice for f_2 is $f_2(x) = -|x|$: if a jump in price is large, so will be $P_{(-,+)}$. With these assumptions, we have

$$m_{t+1} = \frac{m_t + 1}{2} \tanh \beta(Jm_t - |\Delta p_t|) + \frac{1 - m_t}{2} \tanh \beta(Jm_t + |\Delta p_t|). \quad (7)$$

Remarks. Since it is always the case that $P_{(+,+)} \geq P_{(-,+)}$, in the parlance of [11] the model (7) is antihysteretic. Furthermore note that the model no longer has the skew-symmetric

structure as in (1)–(2). In other words, the price p_t is not slaved to the sentiment m_t and so, for example, bimodal dynamics of m_t no longer imposes a bimodal distribution on p_t .

The resulting model (3),(7) has rich dynamics and requires some parameter tuning to exhibit stylized market facts. In particular two highly unrealistic effects occur for large parameter regimes. Firstly, highly oscillatory behaviour can occur where the sentiment displays period-two behaviour with a large fraction of the agents switching every time period. Such a jittery population of agents would quickly be wiped out by transaction costs. The other is the presence of a fixed point at $m_t \equiv 0$ which can be stable for a wide range of parameters. The stability analysis for $\kappa = 0$ is straightforward (see also [4]). Briefly, if we consider

$$m_{t+1} = \frac{m_t + 1}{2} \tanh \beta(Jm_t - s) + \frac{1 - m_t}{2} \tanh \beta(Jm_t + s) \quad (8)$$

with $s > 0$, it is not hard to check that $m_t = 0$ is asymptotically stable if

$$s > \frac{1}{\beta} \tanh^{-1} \left(1 - \frac{1}{\beta J} \right).$$

Now suppose $s = s_t = |\eta_t|$. Then using Maple we obtain

$$\mathbb{E}(s) = \frac{2^{1/2}}{\pi^{1/2}} \sigma.$$

Assuming that it is the expectation of h_t that plays the important rôle, we thus have the following rough statement: $m_t = 0$ is unstable as long as

$$\sigma < \frac{\pi^{1/2}}{2^{1/2}} \frac{1}{\beta} \tanh^{-1} \left(1 - \frac{1}{\beta J} \right).$$

In the next section we shall see that in parameter regimes where both the period-2 cycles and the trivial fixed point are unstable, price/sentiment dynamics can be compared with the stylized facts.

3. Numerics

First we choose the variance of the external information stream η_t since this determines the timescale corresponding to each timestep. For many real-world financial markets the daily standard deviation in prices is approximately 0.5–0.6% and this corresponds to $\text{var}(\eta) = 0.00004$. The liquidity parameter κ is chosen to be 0.12 (similar to the value used in [7]), $J = 0.05$ and $\beta = 40$. A typical simulation is shown in Figure 1 over 10000 timesteps which equates to approximately 40 years of trading.

Figure 1a) plots the price p_t (solid line) and the ‘fundamental’ price p_t^F (dashed line) which is defined by either setting $\kappa = 0$ in (3) or, equivalently, by supposing that the sentiment is always 0. In other words it is the price obtained when the market sentiment is always neutral and the exogenous noise is correctly incorporated into the market price. The fundamental price does not appear directly in the model but when plotted with p_t in Figure 1a), it allows us to see the effect of the sentiment changes on the price dynamics. As expected, the actual price is more variable than the fundamental one and significant deviations can last several years.

In Figure 1b), the sentiment is plotted. Polarized markets, both positive and negative, can remain for significant periods before the sentiment suddenly switches. There are also long intervals where the sentiment fluctuates about 0 without major price booms or crashes forming. Thus the two variables in the model appear to be behaving reasonably, at least in a crude

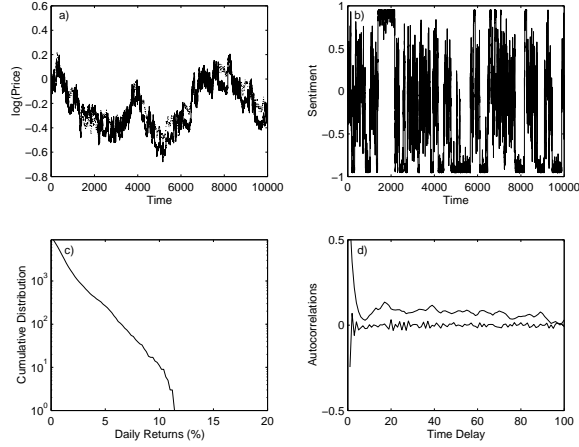


Figure 1. Numerical simulation over a 40 year period. See the text for full details.

qualitative sense, and so we now look more closely at the stylized facts and the ability of the model to reproduce them.

In Figure 1c) we plot the cumulative distribution of the absolute price changes (the number of days on which the absolute percentage price change exceeds a given value). On a semi-log scale a Gaussian distribution of returns would be approximately a straight line. However, there is a clear excess of days on which the return is greater than approximately 3% (in good agreement with statistical data and HAMs). Thus the fat-tail phenomenon is present and this is also reflected in the kurtosis which fluctuates significantly between runs but averages at approximately 9, which is also consistent with measurements from real markets if a little on the low side.

We now examine the autocorrelations present in the price data. Figure 1d) shows two sets of data. The upper curve, exhibiting a very slow decay, is the autocorrelation of the volatility which is defined as the absolute value of the daily price returns. This is indicative of volatility clustering and a time series of the price returns (Figure 2) confirms this. That the model can reproduce this effect is perhaps the most surprising aspect of the simulation. In real markets volatility clustering is often assumed to be due to ‘memory effects’, especially since the Hurst exponent of the volatility time-series is usually much greater than 0.5, suggesting persistence in the data. The model being simulated here has minimal memory, since the sentiment equation can be written as $m_{t+1} = F(m_t, m_{t-1}, \eta_t)$. The volatility autocorrelations arise because the average absolute change in sentiment is itself a function of the sentiment. Close to $m_t = 0$ the absolute change in sentiment over any timestep is very small (because of the multiplying factor m_t on the RHS of (7)) but this is not the case away from $m_t = 0$. Since the price changes themselves depend in part upon the sentiment changes, this induces volatility clustering in the price data. A more general observation [7] is that volatility clustering will occur whenever the average magnitude of price changes is dependent upon a more-slowly varying quantity such as the sentiment. Furthermore there are good economic arguments for supposing that this is indeed the case. For example, in a highly-polarized market, especially a bull market, the presence of extra market participants such as day-traders will increase the volatility until the bubble deflates and sentiment returns to more normal levels. However, to date there appears to have been very little research into the relationship between volatility and sentiment. Such investigations should help clarify the origins of volatility clustering in real markets.

Finally, the lower data-set in Figure 1d) shows the autocorrelation of the daily price-changes themselves (rather than their absolute value). Both the EMH and the stylized facts conclude that autocorrelations in price data vanish over all but the very shortest timescales. However

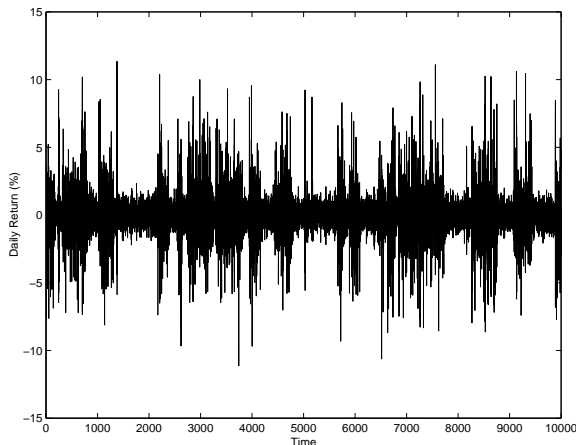


Figure 2. The price return data for the simulation in Figure 1.

we see a noticeable (and therefore exploitable) negative correlation between price changes on adjacent days. This appears to be an artifact of the discretized modeling process and could be mitigated by, for example, choosing the variance of the external noise to correspond to a time-period smaller than the one of interest and then recovering the daily/weekly/monthly price data as required.

4. Conclusions

There are many heterogeneous agent models in the economics and econophysics literature that seem to be capable of reproducing the most important stylized facts of real market data. It has been claimed that the heterogeneity of the agents is necessary for realistic model behaviour.

The model introduced in this paper is not intended to be a replacement for HAMs. The questions concerning when and how trading rules at the micro-level percolate through to global statistics and structures are profoundly important ones. Furthermore, HAMs appear to be more capable of accurately preserving important but short-scale properties (such as the non-correlation of price changes over all but very short timescales) over a very wide range of parameters. However, small, fast models that can quickly produce plausible market-pricing data for long simulation times may be sufficient for many financial purposes and yield important analytical insights.

Furthermore, the model (3), (7) clearly shows the necessity of the two psychological mechanisms postulated in [7]: both the herding (cowardice) effects expressed in the relationships $\partial P_{(+,+)}(x,y)/\partial x > 0$ and $\partial P_{(-,+)}(x,y)/\partial x > 0$ and the inaction effects ($\partial P_{(+,+)}(x,y)/\partial y < 0$ and $\partial P_{(-,+)}(x,y)/\partial y > 0$) are required for the correct prediction of financial times series properties. Thus, it makes sense to consider the model introduced here as in some sense a minimal model capable of generating stylized market facts.

It is also worth noting that the model (3), (7) can be considered as a return of representative agent economics, with m_t being then interpreted as the agent's expected market position. The lesson here is that in order to obtain realistic price return statistics, such an agent must be based on plausible psychological principles and has to follow stochastic behavioral rules.

Acknowledgments

MG would like to thank Prof. O. Penrose for clarifying the legitimacy of passing to the mean field limit and Dr. G. Berkolaiko for stimulating discussions.

References

- [1] Alfarano S and Lux T 2003 A minimal noise trader model with realistic time series properties *Economics Working Paper No. 2003-15, Department of Economics, Christian-Albrechts-Universität, Kiel*
- [2] Arnold L 1998 *Random Dynamical Systems* (Berlin: Springer)
- [3] Bachelier L 1900 *The theory of speculation* reprinted in 1964 *The Random Character of Stock Market Prices* ed P H Cootner (Cambridge: MIT Press)
- [4] Brock W A and Durlauf S N 2001 Discrete choice with social interactions *Review of Economic Studies* **68** pp 235–260
- [5] Camerer C, Loewenstein G and Prelec D 2005 Neuroeconomics *Journal of Economic Literature* **43** pp 9–64
- [6] Campbell J Y, Lo A W and MacKinlay A C 1997 *The Econometrics of Financial Markets* (Princeton: UP)
- [7] Cross R, Grinfeld M, Lamba H and Seaman T 2005 A threshold model of investor psychology *Physica A* **354** pp 463–478
- [8] Daniels M G, Farmer J D, Gillemot L, Iori G and Smith E 2003 Quantitative model of price diffusion and market friction based on trading as a mechanistic random process *Physics Review Letters* **90** pp 108102-1–4
- [9] Fama E F 1970 Efficient capital markets: a review of theory and empirical work *Journal of Finance* **25** pp 283–417
- [10] Farmer J D 1999 Physicists attempt to scale the ivory towers of finance *Computing in Science and Engineering* **1** pp 26–39
- [11] Grinfeld M and Piscitelli L and Cross R 2000 A probabilistic framework for hysteresis *Physica A* **287** pp 577–586
- [12] Hommes C H 2006 Heterogeneous agent models in economics and finance, in: *Handbook of Computational Economics, Volume 2: Agent-Based Computational Economics* eds L Tesfatsion and K L Judd (Amsterdam: Elsevier) pp 1–66
- [13] Horst U 1999 Ergodic fluctuations in a stock market model with interacting agents: the mean field case *Discussion paper No. 106, Sonderforschungsbereich 373, Humboldt Universität, Berlin*
- [14] Keynes J M 1936 *The General Theory of Employment, Interest, and Money* (London: Macmillan)
- [15] Malkiel B G 2003 The efficient markets hypothesis and its critics *Journal of Economic Perspectives* **17** pp 59–82
- [16] Merton R and Brodie Z 2005 Design of financial systems *Journal of Investment Management* **3** (2005) pp 6–28
- [17] Shiller R J 2003 From efficient markets to behavioural finance *Journal of Economic Perspectives* **17** pp 83–104
- [18] Shiller R J 1981 Do stock prices move too much to be justified by subsequent changes in dividends? *American Economic Review* **71** pp 421–436
- [19] Shleifer A and Summers L H 1990 The noise trader approach to finance *Journal of Economic Perspectives* **4** pp 19–33
- [20] Stoll H R 2006 Electronic trading in stock markets *Journal of Economic Perspectives* **20** pp 153–174