AN ADAPTIVE EULER-MARUYAMA SCHEME FOR SDES: CONVERGENCE AND STABILITY

H. LAMBA[1], J.C. MATTINGLY[2] AND A.M. STUART[3]

Abstract. The understanding of adaptive algorithms for SDEs is an open area where many issues related to both convergence and stability (long time behaviour) of algorithms are unresolved. This paper considers a very simple adaptive algorithm, based on controlling only the drift component of a time-step. Both convergence and stability are studied.

The primary issue in the convergence analysis is that the adaptive method does not necessarily drive the time-steps to zero with the user-input tolerance. This possibility must be quantified and shown to have low probability.

The primary issue in the stability analysis is ergodicity. It is assumed that the noise is non-degenerate, so that the diffusion process is elliptic, and the drift is assumed to satisfy a coercivity condition. The SDE is then geometrically ergodic (averages converge to statistical equilibrium exponentially quickly). If the drift is not linearly bounded then explicit fixed time-step approximations, such as the Euler-Maruyama scheme, may fail to be ergodic. In this work, it is shown that the simple adaptive time-stepping strategy cures this problem. In addition to proving ergodicity, an exponential moment bound is also proved, generalizing a result known to hold for the SDE itself.

Key Words: Stochastic Differential Equations, Adaptive Time-Discretization, Convergence, Stability, Ergodicity, Exponential Moment Bounds.

1. Introduction

In this paper, we study the numerical solution of the Itô SDE

$$dx(t) = f(x(t))dt + g(x(t))dW(t), \quad x(0) = X \tag{1.1}$$

by means of an adaptive time-stepping algorithm. Here $x(t) \in \mathbb{R}^m$ for each t and $W(t)$ is a $d-$dimensional Brownian motion. Thus $f : \mathbb{R}^m \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^{m\times d}$. For simplicity we assume that the initial condition is deterministic. Throughout $|\cdot|$ is used to denote either the Euclidean vector norm or the Frobenius (trace) matrix norm as appropriate. We assume throughout that f, g are C^2. Further structural assumptions will be made where needed. The basic adaptive mechanism we study is detailed at the start of the next section. It is a simple adaptive algorithm, prototypical of a whole class of methods for the adaptive integration of SDEs. Our aim is twofold. First we show convergence, as the user-input tolerance τ tends to zero; this is a non-trivial exercise because the adaptive strategy does not imply that the

[1]Department of Mathematical Sciences, George Mason University, Fairfax, VA22030, USA.

hlamba@gmu.edu

[2]Department of Mathematics, Duke University, Durham, NC 27708, USA.

jonm@math.duke.edu

[3]Mathematics Institute, Warwick University, Coventry, CV4 7AL, England.

andrew.m.stuart@warwick.ac.uk

time-steps taken tend to zero with the tolerance everywhere in phase space. Secondly we show that the methods have a variety of desirable properties for the long-time integration of ergodic SDEs, including preservation of ergodicity and exponential moment bounds.

The adaptive method controls the time-step of a forward Euler drift step so that it deviates only slightly from a backward Euler step. This not only controls an estimate of the contribution to the time-stepping error from the drift step, but also allows the analysis of stability (large time) properties for implicit backward Euler methods to be employed in the explicit adaptive method. Numerical experiments suggest that both the convergence and stability analyses extend to a number of more sophisticated methods which control different error measures; some of these experiments are reported below.

It is of interest to discuss our work in the context of a sequence of interesting papers which study the optimality of adaptive schemes for SDEs, using various different error measures [12, 13, 14, 24]. For many of these error measures, which are quite natural in practice, the asymptotically optimal adaptive schemes are based solely on the diffusion. This is essentially because it is the diffusion term which dominates the (lack of) regularity in paths and this regularity in turn dominates error measures. Why then, have we concentrated on methods which adapt only on the drift? The reason for this is that, as mentioned above, such methods are advantageous for long-time integration. In practice, we anticipate that error controls based on both drift and diffusion could combine the advantages of the asymptotically optimal schemes with the enhanced stability/ergodicity of schemes which control based on the drift.

In order to prove a strong mean-square convergence result for this algorithm, it is first necessary to obtain a suitable upper bound on the sequence of timesteps used. These bounds mimic those used in the convergence proofs for adaptive ODE solvers [29, 17, 16] and require that the numerical solution does not enter neighbourhoods of points where the local error estimate vanishes. (Requiring that these neighbourhoods are small excludes some simple drift vector fields, such as constants. In practice, we would anticipate controlling on both the drift and the diffusion, minimizing this issue). An essential part of the analysis is a proof that the contribution to the mean-square error from paths that violate this condition is suitably small.

Adaptivity is widely used in the solution of ordinary differential equations (ODEs) in an attempt to optimize effort expended per unit of accuracy. The adaptation strategy can be viewed heuristically as a fixed time-step algorithm applied to a time re-scaled differential equation [6] and it is of interest to study convergence of the algorithms as the tolerance employed to control adaptation is reduced to zero [17]. However adaptation also confers stability on algorithms constructed from explicit time-integrators, resulting in better qualitative behavior than for fixed time-step counter-parts. This viewpoint was articulated explicitly in [27] and subsequently pursued in [1], [10] and [31] for example. In particular the reference [31] studies the effect of time-discretization on dissipative structures such as those highlighted in [7, 35]. It is shown that certain adaptive strategies have the desirable property of constraining time-steps of explicit integrators so that the resulting solution update differs in a controlled way from an implicit method. Since many implicit methods have desirable stability properties (see [3] and [30], Chapter 5) this viewpoint can be used to facilitate analysis of the stability of adaptive algorithms [31].

In [21], stochastic differential equations (SDEs) with additive noise and vector fields satisfying the dissipativity structures of [7, 35] are studied. There, and in [25, 33, 34], it is shown that explicit time-integrators such as Euler-Maruyama may fail to be ergodic even when the underlying SDE is geometrically ergodic. The reason is that the (mean) dissipativity induced by the drift is lost under time-discretization. Since this is exactly the issue arising for explicit integration of dissipative ODEs, and since this issue can be resolved in that context by means of adaptation, it is natural to study how such adaptive methods impact the ergodicity of explicit methods for SDEs. In recent years, the numerical solution of SDEs with gradient drift vector fields has been used as the proposal for an MCMC method for sampling from a prescribed density, known only up to a multiplicative constant – a technique referred to as Metropolis-adjusted Langevin algorithm [2]. In this context, it is very desirable that the time discretization inherit ergodicity. The adaptive scheme proposed here is an approach to ensuring this. In this sense our work complements a number of recent papers concerned with constructing approximation schemes which are ergodic in situations where the standard fixed step Euler-Maruyama scheme fails to be: in [25] a Metropolis-Hastings rejection criterion is used to enforce ergodicity; in [8, 28] local linearization is used; in [21] implicit methods are used. Although the adaptive method that we analyze here is proved to be convergent on finite time intervals, it would also be of interest to extend the work of Talay [32], concerned with convergence proofs for invariant measure under time-discretization, to the adaptive time-step setting considered here.

In Section 2, we introduce the adaptive algorithm, together with some notation. In Section 3 the finite time convergence result for the adaptive method is stated. The proof is given in Section 4 and proceeds by extending the fixed step proof given in [11]; the extension is non-trivial because the adaptivity does not force time-steps to zero with the tolerance in all parts of the phase space. In Section 5, we state the main results of the paper on the stability of the adaptive method. All results are proved under the dissipativity condition

$$\exists \alpha, \beta \in (0,\infty) : \langle f(x), x\rangle \le \alpha - \beta |x|^2 \quad \forall x \in \mathbb{R}^m, \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ is the inner-product inducing the Euclidean norm, as well as a boundedness and invertibility condition on the diffusion matrix g. The results proven include ergodicity and an exponential moment bound; all mimic known results about the SDE itself under (1.2). Section 6 starts with a number of *a priori* estimates for the adaptive scheme of Section 2, and proceeds to proofs of the stability stated in Section 5. Numerical results studying both convergence and ergodicity are presented in Sections 7–9. Some concluding remarks and generalizations are given in Section 10.

2. Algorithm

The adaptive algorithm for (1.1) is as follows:

$$\begin{aligned} k_n &= \mathcal{G}(x_n, k_{n-1}), \quad k_{-1} = K \\ x_{n+1} &= \mathcal{H}(x_n, \Delta_n) + \sqrt{\Delta_n} g(x_n)\eta_{n+1}, \quad x_0 = X, \end{aligned}$$

where $\Delta_n = 2^{-k_n}\Delta_{max}$. Here

$$\mathcal{H}(x,t) = x + tf(x)$$

and

$$\mathcal{G}(x,l) = \min\{k \in \mathbb{Z}^+ : |f(\mathcal{H}(x, 2^{-k}\Delta_{max})) - f(x)| \le \tau \ \& \ k \ge l-1\}.$$

The random variables $\eta_j \in \mathbb{R}^d$ form an i.i.d. sequence distributed as $\mathcal{N}(0, I)$. The parameter K defines the initial time-step and $\tau > 0$ the tolerance. Note that the algorithm defines a Markov chain for (x_n, k_{n-1}) on $\mathbb{R}^d \times \mathbb{Z}^+$.

We may write

$$\begin{aligned} x_n^\star &= x_n + \Delta_n f(x_n), \\ x_{n+1} &= x_n^\star + \sqrt{\Delta_n} g(x_n)\eta_{n+1}. \end{aligned} \tag{2.1}$$

If $K \in \mathbb{Z}^+$ then $k_n \in \mathbb{Z}^+$ and the error control enforces the condition

$$\Delta_n \leq \min\{2\Delta_{n-1}, \Delta_{\max}\},$$

where Δ_{max} is the fixed maximum time-step. Furthermore we have

$$|f(x_n^\star) - f(x_n)| \leq \tau.$$

In the absence of noise, this implies that the difference between an Euler approximation at the next time-step, and an explicit second order approximation, is of size $\mathcal{O}(\Delta_n \tau)$. In the presence of noise, it imposes a similar restriction on the means. As mentioned in the introduction, in practice we would anticipate combining this drift error control with others tuned to the diffusion.

2.1. **Notation.** The most important notation conceptually is concerned with making relationships between the numerical approximations at discrete steps, and the true solution at certain points in time. To do this we define $\mathcal{F}_n$ to be the sigma-algebra generated by n steps of the Markov chain for (x_n, k_{n-1}). Let

$$t_n = t_{n-1} + \Delta_{n-1}, \quad t_0 = 0,$$

$\delta > 0$ and define the stopping times N_j by $N_0 = 0$ and, for $j \geq 1$,

$$N_j = \inf_{n \geq 0}\{n : t_n \geq \delta + t_{N_{j-1}}\}.$$

Where the dependence on δ is important we will write $N_j(\delta)$. It is natural to examine the approximate process at these stopping times since they are spaced approximately at fixed times in the time variable t. Theorem 2 in Section 5 shows that these stopping times are almost surely finite, under the dissipativity condition (1.2). Notice that

$$\delta^- := \delta \leq t_{N_j} - t_{N_{j-1}} \leq \delta + \Delta_{max} := \delta^+.$$

When considering strong convergence results it is necessary to interpret $\sqrt{\Delta_n}\eta_{n+1}$ in the adaptive algorithm as the Brownian increment $W(t_{n+1}) - W(t_n)$.

We let

$$y_j = x_{N_j+1} \quad \text{and} \quad l_j = k_{N_j}.$$

The Markov chain for $\{y_j, l_j\}$ will be important in our study of long time behaviour and we will prove that it is ergodic. Let $\mathcal{G}_j = \mathcal{F}_{N_j}$, the filtration of events up to the j^{th} stopping time.

It is convenient to define two continuous time interpolants of the numerical solution. We set

$$X(t) = x_n, \quad t \in [t_n, t_{n+1}), \tag{2.2}$$

$$\overline{X}(t) = X + \int_0^t f(X(s))ds + \int_0^t g(X(s))dW(s). \tag{2.3}$$

Hence, for $t \in [t_n, t_{n+1})$

$$\overline{X}(t) = x_n + (t - t_n)f(x_n) + g(x_n)[W(t) - W(t_n)] \tag{2.4}$$

$$= (1 - \alpha_n(t))x_n + \alpha_n(t)x_n^\star + g(x_n)[W(t) - W(t_n)] \tag{2.5}$$

for $\alpha_n(t) = (t - t_n)/(t_{n+1} - t_n) \in [0, 1)$.

It is sometimes important to know the smallest step-size beneath which the error control is always satisfied, at a given point x. Hence we define

$$k^\star(x) = \min\{k \in \mathbb{Z}^+ : |f(\mathcal{H}(x, 2^{-l}\Delta_{max})) - f(x)| \leq \tau \; \forall l \geq k\} \quad \text{and} \quad k^\star(B) = \sup_{x \in B} k^\star(x),$$

noting that, by continuity of f, $k^\star(B)$ is finite if B is bounded.

Because of the boundedness of g we deduce that there are functions $\sigma(x)$ and constants $\sigma, a > 0$ such that, for η distributed as η_1 and independent of x,

$$\mathbb{E}|g(x)\eta|^2 := \sigma^2(x) \le \sigma^2, \quad \text{and} \quad \mathbb{E}\big||g(x)\eta|^2 - \sigma^2(x)|^2\big| \le a\sigma^4.$$

The following definitions will be useful:

$$\tilde{\alpha} = \alpha + \frac{1}{2}\tau, \quad \tilde{\beta} = \beta - \frac{1}{2}\tau, \quad \beta_n = \frac{1}{1+2\tilde{\beta}\Delta_n}, \quad \bar{\gamma} = 1 + \tilde{\beta}\Delta_{max}.$$

We will always assume that τ is chosen small enough so that $\tilde{\beta} > 0$. The constants γ^- is chosen so that

$$(1+t)^{-1} \le (1-\gamma^- t) \le e^{-\gamma^- t} \quad \forall t \in [0, 2\tilde{\beta}\Delta_{max}].$$

3. Convergence Result

We start by discussing the error control mechanism. We define $F_1(u)$ by

$$F_1(u) = df(u)f(u),$$

the function $F_2(u,h)$ by

$$F_2(u,h) := h^{-1}\Big(f(u+hf(u)) - f(u) - hF_1(u)\Big)$$

and $E(u,h)$ by

$$E(u,h) = f(u+hf(u)) - f(u).$$

Now, since $f \in C^2$, Taylor series expansion gives

$$E(u,h) = h[F_1(u) + hF_2(u,h)] \tag{3.6}$$

where F_1, F_2 are defined above. Note that the error control forces a time-step so that the norm of $E(x_n, \Delta_n)$ is of order $\mathcal{O}(\tau)$. Estimating the implications of this for the time-step Δ_n forms the heart of the convergence proof below.

In order to state the assumptions required for the convergence result we define, for $R, \epsilon \ge 0$, the sets

$$\Psi(\epsilon) = \{u \in \mathbb{R}^m : |F_1(u)| \le \epsilon\}, \;\; B_R = \{u \in \mathbb{R}^m : |u| \le R\} \;\text{ and }\; B_{R,\epsilon} = B_R \setminus \Psi(\epsilon)$$

and introduce the constant $K_R = \sup_{u\in B_R, h\in[0,\Delta t_{\max}]} |F_2(u,h)|$. Now define the following:

$$\begin{array}{lll} \sigma_R := \inf\{t \ge 0 : |\overline{X}(t)| \ge R\}, & \rho_R := \inf\{t \ge 0 : |x(t)| \ge R\}, & \theta_R := \sigma_R \wedge \rho_R \\ \sigma_\epsilon := \inf\{t \ge 0 : |F_1(\overline{X}(t))| \le \epsilon\}, & \rho_\epsilon := \inf\{t \ge 0 : |F_1(x(t))| \le 2\epsilon\}, & \theta_\epsilon := \sigma_\epsilon \wedge \rho_\epsilon \\ \sigma_{R,\epsilon} := \sigma_R \wedge \sigma_\epsilon, & \rho_{R,\epsilon} := \rho_R \wedge \rho_\epsilon, & \theta_{R,\epsilon} := \theta_R \wedge \theta_\epsilon. \end{array}$$

The first assumption is a local Lipschitz condition on the drift and diffusion co-efficients, together with moment bounds on the true and numerical solutions.

Assumption 3.1. *For each $R > 0$ there exists a constant C_R, depending only on R, such that*

$$|f(a) - f(b)|^2 \vee |g(a) - g(b)|^2 \le C_R|a-b|^2, \forall a, b \in \mathbb{R}^m \text{ with } |a| \vee |b| \le R. \tag{3.7}$$

For some $p > 2$ there is a constant A, uniform in $\tau \to 0$, such that

$$\mathbb{E}\left[\sup_{0\le t\le T} |\overline{X}(t)|^p\right] \vee \mathbb{E}\left[\sup_{0\le t\le T} |x(t)|^p\right] \le A. \tag{3.8}$$

Note that inequality (3.7) is a local Lipschitz assumption which will be satisfied for any f and g in C^2. The inequality (3.8) states that the p^{th} moments of the exact and numerical solution are bounded for some $p > 2$. Theorem 4 proves (3.8) for the numerical interpolant, under natural assumption on f and g (see Assumption 5.1). Under the same assumptions, such a bound is known to hold for $x(t)$; see [19].

We clearly also need an assumption on the local error estimate since if, for example, the drift term $f(u)$ were constant then $E(u, h) \equiv 0$ and the stepsize would, through doubling, reach $\Delta_{\max}$, no matter how small τ is, and convergence cannot occur as $\tau \to 0$. Because the function $F_1(u)$ maps $\mathbb{R}^m$ into itself, the following assumption on the zeros of $F_1(u)$ will hold for generic drift functions f which are non-constant on any open sets; it does exclude, however, the case of constant drift. Furthermore the assumption on the hitting time rules out dimension $m = 1$.

Assumption 3.2. *Define*

$$\ell(\epsilon, R) = d_H\{\Psi(2\epsilon)^c \cap B_R, \Psi(\epsilon) \cap B_R\}.$$

For any given $R > 0$ we assume that $\ell(\epsilon, R) > 0$ for all sufficiently small $\epsilon > 0$, and that $\ell(\epsilon, R) \to 0$ as $\epsilon \to 0$. Furthermore, the hitting time ρ_ϵ satisfies, for any $X \notin \Psi(0)$,

$$\rho_\epsilon \to \infty \text{ as } \epsilon \to 0 \text{ a.s.}$$

Here d_H denotes Hausdorff distance. The preceding assumption requires that the contours defining the boundary of $\Psi(\epsilon)$ are strictly nested as ϵ increases, and bounded. This enables us to show that the probability of $(x(t), \overline{X}(t)) \in (\Psi(2\epsilon)^c \cap B_R) \times (\Psi(\epsilon) \cap B_R)$ is small, a key ingredient in the proof.

We now state the strong convergence of the adaptive numerical method, using the continuous-time interpolant $\overline{X}(t)$. Note that we do not assume $\Delta_{max} \to 0$ for this theorem. Hence the non-standard part of the proof comes from estimating the contribution to the error from regions of phase space where the time-step is not necessarily small as $\tau \to 0$.

Theorem 1. *Assume that $X \notin \Psi(0)$. Let Assumptions 3.1 and 3.2 hold. Then, there is $\Delta_c(\tau)$ such that, for all $\Delta_{-1} < \Delta_c(\tau)$ and any $T > 0$, the numerical solution with continuous-time extension $\overline{X}(t)$ satisfies*

$$\mathbb{E}\left[\sup_{0\le t\le T} |\overline{X}(t) - x(t)|^2\right] \to 0 \quad as \quad \tau \to 0.$$

4. Proof of Convergence Result

The primary technical difficulty to address in convergence proofs for adaptive methods is to relate the time-step to the tolerance τ. Roughly speaking the formula (3.6) shows that, provided $F_1(u) \neq 0$, the error control will imply $\Delta_n = \mathcal{O}(\tau)$. We now make this precise. We provide an upper bound on the timestep sequence of numerical solutions that remain within $B_{R,\epsilon}$, for sufficiently small τ. For given $R, \epsilon > 0$ we define the quantities

$$\overline{h}_{R,\epsilon} = \frac{\epsilon}{6K_R} \quad \text{and} \quad \tau_{R,\epsilon} = \frac{\epsilon^2}{12K_R}.$$

Lemma 4.1. *For any $R, \epsilon > 0$, if $\{x_n\}_{n=0}^N \subseteq B_{R,\epsilon}$, $\tau < \tau_{R,\epsilon}$ and $\Delta_{-1} < \frac{2\tau}{\epsilon}$ then*

$$\Delta_n \le \min\{\overline{h}_{R,\epsilon}, \frac{2\tau}{\epsilon}\} \quad \forall\, 0 \le n \le N. \tag{4.9}$$

Proof. The error control implies

$$|E(x_n,\Delta_n)| = \Delta_n\,|F_1(x_n)+\Delta_n F_2(x_n,\Delta_n)| \le \tau.$$

Note that

$$\Delta_{-1} < 2\tau_{R,\epsilon}/\epsilon = \overline{h}_{R,\epsilon}.$$

We first proceed by contradiction to prove $\Delta_n \le \overline{h}_{R,\epsilon}\ \ \forall\, 0\le n\le N$. Let $0\le m\le N$ be the first integer such that $\Delta_m > \overline{h}_{R,\epsilon}$. Then, since there is a maximum timestep ratio of 2, we have

$$\begin{aligned}\Delta_m \in (\frac{\epsilon}{6K_R},\frac{\epsilon}{3K_R}] \ \ &\Rightarrow\ \ \Delta_m|F_2(x_m,\Delta_m)| < \frac{\epsilon}{2}\\ &\Rightarrow\ \ |E(x_m,\Delta_m)| > \Delta_m(\epsilon-\epsilon/2) \ge \frac{\epsilon\overline{h}_{R,\epsilon}}{2} = \frac{\epsilon^2}{12K_R} = \tau_{R,\epsilon} > \tau.\end{aligned}$$

Thus Δ_m is not an acceptable timestep, contradicting our original assumption. The first result follows. The proof of the bound on the timestep in (4.9) now follows immediately since

$$\Delta_n \le \frac{\tau}{|F_1(x_n)+\Delta_n F_2(x_n,\Delta_n)|} \le \frac{\tau}{(\epsilon-\epsilon/2)} \le \frac{2\tau}{\epsilon} \quad \forall\, 0\le n\le N.$$

□

Proof of Theorem 1 We denote the error by

$$e(t) := \overline{X}(t) - x(t).$$

Recall the Young inequality: for $r^{-1}+q^{-1}=1$

$$ab \le \frac{\delta}{r}a^r + \frac{1}{q\delta^{q/r}}b^q, \quad \forall a,b,\delta>0.$$

We thus have for any $\delta>0$

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2\right] &= \mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2\mathbf{1}\{\theta_{R,\epsilon}>T\}\right] + \mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^2\mathbf{1}\{\theta_{R,\epsilon}\le T\}\right]\\ &\le \mathbb{E}\left[\sup_{0\le t\le T}|e(t\wedge\theta_{R,\epsilon})|^2\mathbf{1}\{\theta_{R,\epsilon}>T\}\right] + \frac{2\delta}{p}\mathbb{E}\left[\sup_{0\le t\le T}|e(t)|^p\right]\\ &\quad + \frac{1-\frac{2}{p}}{\delta^{2/(p-2)}}\mathbb{P}\big(\theta_{R,\epsilon}\le T\big).\end{aligned} \tag{4.10}$$

Now

$$\mathbb{P}\left(\theta_{R,\epsilon}\le T\right) = \mathbb{P}\{\theta_R\le T\} + \mathbb{P}\{\theta_\epsilon\le T,\theta_R>T\}.$$

But

$$\mathbb{P}\{\theta_R\le T\} \le \mathbb{P}\{\sigma_R\le T\} + \mathbb{P}\{\rho_R\le T\}$$

whilst

$$\mathbb{P}\{\theta_\epsilon\le T,\theta_R>T\} \le \mathbb{P}\{\rho_\epsilon\le T\} + \mathbb{P}\{\theta_\epsilon\le T,\theta_R>T,\rho_\epsilon>T\}.$$

Thus we have

$$\mathbb{P}\left(\theta_{R,\epsilon}\le T\right) \le \mathbb{P}(\sigma_R\le T) + \mathbb{P}(\rho_R\le T) + \mathbb{P}(\rho_\epsilon\le T) + \mathbb{P}\{\theta_\epsilon\le T,\theta_R>T,\rho_\epsilon>T\}.$$

To control the last term notice that whenever $\theta_\epsilon\le T$, $\theta_R>T$ and $\rho_\epsilon>T$ we know that $|e(\sigma_\epsilon)|\ge \ell(\epsilon,R)$. Hence we have

$$\mathbb{P}\{\theta_\epsilon\le T,\theta_R>T,\rho_\epsilon>T\} \le \mathbb{P}\{|e(T\wedge\theta_{R,\epsilon})|\ge\ell(\epsilon,R)\} \le \mathbb{E}|e(T\wedge\theta_{R,\epsilon})|^2/\ell(\epsilon,R)^2.$$

Combining the two preceding inequalities gives

$$\mathbb{P}\left(\theta_{R,\epsilon} \le T\right) \le \mathbb{P}(\sigma_R \le T) + \mathbb{P}(\rho_R \le T) + \mathbb{P}(\rho_\epsilon \le T) + \mathbb{E}\Big(\sup_{0\le t\le T} |e(t\wedge\theta_{R,\epsilon})|^2\Big)/\ell(\epsilon,R)^2.$$

By Markov's inequality

$$\mathbb{P}\{\sigma_R \le T\}, \mathbb{P}\{\rho_R \le T\} \le \frac{A}{R^p}.$$

so that

$$\mathbb{P}\left(\theta_{R,\epsilon} \le T\right) \le \frac{2A}{R^p} + \mathbb{P}(\rho_\epsilon \le T) + \mathbb{E}\Big(\sup_{0\le T} |e(t\wedge\theta_{R,\epsilon})|^2\Big)/\ell(\epsilon,R)^2. \tag{4.11}$$

Furthermore,

$$\mathbb{E}\left[\sup_{0\le t\le T} |e(t)|^p\right] \le 2^{p-1}\mathbb{E}\left[\sup_{0\le t\le T}\left(|\overline{X}(t)|^p + |x(t)|^p\right)\right] \le 2^p A. \tag{4.12}$$

Using (4.11), (4.12) in (4.10) gives, for ϵ sufficiently small,

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T} |e(t)|^2\right] \quad &\le \quad \left(1+\frac{p-2}{p\delta^{2/(p-2)}\ell(\epsilon,R)^2}\right)\mathbb{E}\left[\sup_{0\le t\le T}|e(t\wedge\theta_{R,\epsilon})|^2\right] \\ &\quad + \frac{2^{p+1}\delta A}{p} + \frac{(p-2)}{p\delta^{2/(p-2)}}\left[\frac{2A}{R^p} + \mathbb{P}\{\rho_\epsilon \le T\}\right].\end{aligned} \tag{4.13}$$

Take any $\kappa > 0$. To complete the proof we choose δ sufficiently small so that the second term on the right hand side of (4.13) is bounded by $\kappa/4$ and then R and ϵ sufficiently large/small so that the third and fourth terms are bounded by $\kappa/4$. Now reduce τ so that Lemma 4.1 applies. Then, by further reduction of τ in Lemma 4.2, we upper-bound the first term by $\kappa/4$. (Lemma 4.2 calculates the error conditioned on the true and numerical solutions staying within a ball of radius R, and away from small sets where the error control mechanism breaks down. With this conditioning it follows from Lemma 4.1 that we have $\Delta_n = \mathcal{O}(\tau)$, which is the essence of why Lemma 4.2 holds.)

Consequently we have

$$\mathbb{E}\left[\sup_{0\le t\le T} |\overline{X}(t) - x(t)|^2\right] \le \kappa$$

and since κ is arbitrary the required result follows. $\square$

In the following, C is a universal constant independent of T, R, ϵ, δ and τ. Likewise C_R is a universal constant depending upon R, but independent of T, ϵ, δ and τ, $C_{R,T}$ is a universal constant depending upon R and T, but independent of ϵ, δ and τ and $C_{R,\epsilon,T}$ and so forth are defined similarly. The actual values of these constants may change from one occurrence to the next.

Lemma 4.2. *Assume that $X \notin \Psi(0)$ and that τ is sufficiently small for the conditions of Lemma 4.1 to hold. Then the continuous interpolant of the numerical method, $\overline{X}(t)$, satisfies the following error bound:*

$$\mathbb{E}\left[\sup_{0\le t\le T} |\overline{X}(t\wedge\theta_{R,\epsilon}) - x(t\wedge\theta_{R,\epsilon})|^2\right] \le C_{R,\epsilon,T}\tau.$$

Proof. Using

$$x(t\wedge\theta_{R,\epsilon}) := X + \int_0^{t\wedge\theta_{R,\epsilon}} f(x(s))ds + \int_0^{t\wedge\theta_{R,\epsilon}} g(x(s))dW(s),$$

equation (2.3) and Cauchy–Schwartz, we have that $\chi := |\overline{X}(t\wedge\theta_{R,\epsilon}) - x(t\wedge\theta_{R,\epsilon})|^2$, satisfies

$$\begin{aligned}\chi &= \left|\int_0^{t\wedge\theta_{R,\epsilon}} \Big(f(X(s)) - f(x(s))\Big)ds + \int_0^{t\wedge\theta_{R,\epsilon}} \Big(g(X(s)) - g(x(s))\Big)dW(s)\right|^2 \\ &\le 2\left[T\int_0^{t\wedge\theta_{R,\epsilon}} |f(X(s)) - f(x(s))|^2 ds + \left|\int_0^{t\wedge\theta_{R,\epsilon}} \Big(g(X(s)) - g(x(s))\Big)dW(s)\right|^2\right].\end{aligned}$$

Let

$$E(s) := \left[\sup_{0\le t\le s} |\overline{X}(t\wedge\theta_{R,\epsilon}) - x(t\wedge\theta_{R,\epsilon})|^2\right]$$

Then, from the local Lipschitz condition (3.7) and the Doob-Kolmogorov Martingale inequality [26], we have for any $t^* \le T$

$$\begin{aligned}\mathbb{E}E(t^*) &\le 2C_R(T+4)\mathbb{E}\int_0^{t^*\wedge\theta_{R,\epsilon}} |X(s) - x(s)|^2 ds \\ &\le 4C_R(T+4)\mathbb{E}\int_0^{t^*\wedge\theta_{R,\epsilon}} \Big[|X(s) - \overline{X}(s)|^2 + |\overline{X}(s) - x(s)|^2\Big]ds \\ &\le 4C_R(T+4)\left[\mathbb{E}\int_0^{t^*\wedge\theta_{R,\epsilon}} |X(s) - \overline{X}(s)|^2 ds + \int_0^{t^*} \mathbb{E}E(s)ds\right]. \qquad (4.14)\end{aligned}$$

Given $s \in [0, T\wedge\theta_{R,\epsilon})$, let k_s be the integer for which $s \in [t_{k_s}, t_{k_s+1})$. Notice that t_{k_s} is a stopping time because Δ_{k_s} is a deterministic function of $(x_{k_s}, \Delta_{k_s-1})$. We now bound the right hand side in (4.14). From the local Lipschitz condition (3.7), a straightforward calculation shows that

$$|X(s) - \overline{X}(s)|^2 \le C_R(|x_{k_s}|^2 + 1)(\Delta_{k_s}^2 + |W(s) - W(t_{k_s})|^2).$$

Now, for $s < \theta_{R,\epsilon}$, using Lemma 4.1,

$$\begin{aligned}|W(s) - W(t_{k_s})|^2 &= s - t_{k_s} + 2\int_{t_{k_s}}^s [W(l) - W(t_{k_s})]dW(l) \\ &\le (s - t_{k_s})[1 + I(s)] \le \frac{2\tau}{\epsilon}[1 + I(s)].\end{aligned}$$

Here

$$I(s) = \frac{2}{(s - t_{k_s})}\left|\int_{t_{k_s}}^s [W(l) - W(t_{k_s})]dW(l)\right|.$$

Let $\mathcal{H}_s$ denote the $\sigma-$algebra of Brownian paths up to time t_{k_s}. Then, conditioned on $\mathcal{H}_s$, we have

$$\mathbb{E}I(s) \le \sqrt{2}. \qquad (4.15)$$

Thus, using Lemma 4.1, (3.8) and the Lyapunov inequality [15],

$$\begin{aligned}\mathbb{E}\int_0^{t^*\wedge\theta_{R,\epsilon}} |X(s) - \overline{X}(s)|^2 ds &\le \mathbb{E}\int_0^{t^*\wedge\theta_{R,\epsilon}} C_R(|x_{k_s}|^2 + 1)(4\tau^2/\epsilon^2 + |W(s) - W(t_{k_s})|^2)ds \\ &\le C_{R,\epsilon}\tau\mathbb{E}\int_0^{t^*} (1 + |x_{k_s}|^2)(1 + I(s))ds \\ &\le C_{R,\epsilon,T}(A^{2/p+1})\tau.\end{aligned}$$

To obtain the last line we condition on $\mathcal{H}_s$ so that $|x_{k_s}|^2$ and $I(s)$ are independent; we then use (4.15) and the assumed moment bound.

In (4.14), we then have by Lemma 4.1

$$\mathbb{E}E(t^*) \quad \leq \quad C_{R,\epsilon,T}\tau + 4C_{R,T}\int_0^{t^*} \mathbb{E}E(s)ds.$$

Applying the Gronwall inequality we obtain

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left(\overline{X}(t\wedge\theta_{R,\epsilon}) - x(t\wedge\theta_{R,\epsilon})\right)^2\right] \leq C(R,\epsilon,T)\tau.$$

□

5. Stability Results

For all of our stability results, in this and the following sections, we make the assumption that (1.2) holds, together with some conditions on the diffusion matrix. To be explicit we make

Assumption 5.1. *There exists finite positive α, β such that*

$$\langle f(x), x\rangle \leq \alpha - \beta|x|^2 \quad \forall x \in \mathbb{R}^m,$$

where $\langle\cdot,\cdot\rangle$ is the inner-product inducing the Euclidean norm $|\cdot|$. Furthermore $m = d$, g is globally bounded and is globally invertible.

The assumption is made, without explicit statement, for the remainder of the paper. We also assume, without explicit statement, that $\tau < 2\beta$ so that $\tilde{\beta} > 0$. Finally we assume, also without explicit statement, that there is at least one point $\overline{y} \in \mathbb{R}^m$ such that

$$k^*(\overline{y}) = \mathcal{G}(\overline{y}, 1). \tag{5.16}$$

This may implicitly force upper bounds on τ and Δ_{max}, although neither is necessarily restricted by this assumption. The existence of such a $\overline{y}$ is implied by Assumption 5.1, which rules out f being identically constant. Then there exists $\overline{y}$ for which the function

$$|f(\overline{y} + hf(\overline{y})) - f(\overline{y})|$$

is non-zero in a neighbourhood of $h = 0$ and (5.16) must hold, possibly after enforcing bounds on τ and Δ_{max}.

Under Assumption 5.1 the solution of (1.1) exists for all $t > 0$ [9, 19] and the equation is geometrically ergodic [9, 23, 21]. The first stability result ensures that the method will not decrease its stepsize in such a way that it is unable to reach arbitrary finite times.

Theorem 2. *The stopping times N_j are almost surely finite.*

The next result is the main ergodic result of the paper. It ensures that the adaptive method has an attracting statistical steady state. Letting $\mathbb{E}^{y,l}$ denote the expectation under the Markov chain started at $x_0 = y, k_{-1} = l$, we have the following result. (Recall δ occurring in the definition of stopping times N_j.)

Theorem 3. *Assume that $\delta > 5\Delta_{max}$. The Markov chain $\{y_j, l_j\} = \{x_{N_j+1}, k_{N_j}\}$ has a unique invariant measure π. Furthermore, if $h : \mathbb{R}^m \times \mathbb{Z}^+ \to \mathbb{R}$ is measurable and*

$$|h(y,l)| \leq 1 + |y|^2 \quad \forall (y,l) \in \mathbb{R}^m \times \mathbb{Z}^+,$$

then there exists $\lambda \in (0,1)$, $\kappa \in (0,\infty)$ such that

$$|\mathbb{E}^{y_0,l_0}h(y_n,l_n) - \pi(h)| \leq \kappa\lambda^n[1 + |y_0|^2].$$

The final result gives a moment bound on the continuous time interpolants of the numerical solution, mimicking that for the SDE itself.

Theorem 4. *There exists a $\lambda > 0$ and a $c > 0$ so that*

$$\mathbb{E}\exp(\lambda \sup_{t\in[0,T]} \|X(t)\|^2) \leq \exp(\lambda |X|^2 + cT)$$
$$\mathbb{E}\exp(\lambda \sup_{t\in[0,T]} \|\overline{X}(t)\|^2) \leq \exp(\lambda |X|^2 + cT).$$

6. Proof of Stability Results

We start with a number of estimates which will be needed to prove the main results. It is useful to define

$$\xi_{n+1} = 2\sqrt{\Delta_n}\langle x_n^\star, g(x_n)\eta_{n+1}\rangle, \quad \tilde{\xi}_{n+1} = \Delta_n[|g(x_n)\eta_{n+1}|^2 - \sigma^2(x_n)],$$
$$M_n = \sum_{j=0}^{n-1} \xi_{j+1}, \quad \tilde{M}_n = \sum_{j=0}^{n-1} \tilde{\xi}_{j+1}.$$

Observe that $\langle x_n^\star, g(x_n)\eta_{n+1}\rangle$ is a Gaussian random variable conditioned on the values of x_n and $x_n^\star$. Hence the last two expressions are Martingales satisfying the assumptions of Lemma A.1 from the appendix. Also notice that the quadratic variations satisfy

$$\langle M\rangle_n \leq \sum_{j=0}^{n-1} 4\Delta_j |x_j^\star|^2\sigma^2 \quad \text{and} \quad \langle \tilde{M}\rangle_n \leq \sum_{j=0}^{n-1} a\Delta_j^2\sigma^4. \tag{6.1}$$

We start with a straightforward lemma.

Lemma 6.1. *The sequences $\{x_n^\star\}$ and $\{x_n\}$ satisfy*

$$|x_n^\star|^2 \leq |x_n|^2 + 2\Delta_n[\tilde{\alpha} - \tilde{\beta}|x_n^\star|^2],$$
$$|x_{n+1}|^2 \leq \beta_n |x_n|^2 + \Delta_n[2\tilde{\alpha} + \sigma^2] + \xi_{n+1} + \tilde{\xi}_{n+1}.$$

Hence

$$\langle M\rangle_n \leq 4\sigma^2 \sum_{j=0}^{n-1} |x_j|^2\Delta_j + 8\sigma^2\tilde{\alpha}\sum_{j=0}^{n-1}\Delta_j^2.$$

Proof: Taking the inner product of the equation

$$x_n^\star = x_n + \Delta_n f(x_n)$$

with $x_n^\star$ and using the fact that the error control implies

$$|f(x_n^\star) - f(x_n)| \leq \tau,$$

a straightforward calculation from [31], using (1.2), gives the first result. To get the second simply square the expression (2.1) for x_{n+1} and use the first, noting that $\beta_n \leq 1$. For the third use the first in the bound (6.1) for $\langle M\rangle_n$. □

Lemma 6.2. *We have*

$$|x_{n+1}|^2 \leq |X|^2 + C_0 t_{n+1} + M_{n+1} - \frac{1}{2}\frac{\tilde{\beta}}{\sigma^2}\langle M\rangle_{n+1} + \tilde{M}_{n+1} - 2\langle\tilde{M}\rangle_{n+1}\ .$$

where $C_0 = [2\tilde{\alpha} + 4\sigma^4\Delta_{max}]$. Furthermore

$$\mathbb{P}\Big(\sup_{0\leq n}\{|x_n|^2 - C_0 t_n\} \geq |X|^2 + A\Big) \leq 2\exp\big(-BA\big)$$

where B is a positive constant depending only on σ and $\tilde{\beta}$.

Proof: Squaring the expression for x_{n+1} in (2.1), bounding $|x_n^\star|^2$ by the first inequality in Lemma 6.1 and summing gives

$$|x_{n+1}|^2 \le |X|^2 + C_0 t_{n+1} + S_{n+1} + \tilde{S}_{n+1}$$

where

$$S_{n+1} = M_{n+1} - 2\tilde{\beta}\sum_{k=0}^{n} |x_k^\star|^2 \Delta_k, \quad \tilde{S}_{n+1} = \tilde{M}_{n+1} - 4\sigma^4 \Delta_{max} t_{n+1}$$

and M_n, $\tilde{M}_n$ are as before. Using (6.1), one obtains

$$\langle \tilde{M} \rangle_{n+1} \le 2\sigma^4 \Delta_{max} t_{n+1} \ .$$

and

$$\langle M \rangle_{n+1} \le 4\sigma^2 \sum_{k=0}^{n} \Delta_k |x_k^\star|^2 \ .$$

Combining all of this produces,

$$S_{n+1} \le M_{n+1} - \frac{1}{2}\frac{\tilde{\beta}}{\sigma^2}\langle M \rangle_{n+1} \quad \text{and} \quad \tilde{S}_{n+1} \le \tilde{M}_{n+1} - 2\langle \tilde{M} \rangle_{n+1} \ .$$

The probabilistic estimate follows from the exponential martingale estimates from the Appendix. □

Corollary 6.3. *Then there exists a universal $\lambda > 0$ and $C_1 > 0$ so that for any stopping time N with $0 \le t_N \le t_*$ almost surely, for some fixed number t_*, one has*

$$\mathbb{E} \exp(\lambda \sup_{0 \le n \le N} |x_n|^2) \le C_1 \exp(\lambda |X|^2 + \lambda C_0 t_*) \ .$$

Proof. The result follows from Lemma 6.2 and the observation that

$$\mathbb{P}\Big(\sup_{0 \le n \le N} |x_n|^2 \ge |X|^2 + C_0 t_* + A\Big) \le \mathbb{P}\Big(\sup_{0 \le n}\{|x_n|^2 - C_0 t_n\} \ge |X|^2 + A\Big) \ .$$

□

Lemma 6.4. *The Markov chain $\{x_{N_j}\}_{j \in \mathbb{Z}^+}$ satisfies the Foster-Lyapunov drift condition*

$$\mathbb{E}\{|x_{N_{j+1}}|^2 | \mathcal{F}_{N_j}\} \le \exp(-2\gamma^- \tilde{\beta}\delta^-)|x_{N_j}|^2 + \exp(2\tilde{\beta}\delta^+)[2\tilde{\alpha} + \sigma^2]\delta^+.$$

That is

$$\mathbb{E}\{|y_{j+1}|^2 | \mathcal{G}_j\} \le \exp(-2\gamma^- \tilde{\beta}\delta^-)|y_j|^2 + \exp(2\tilde{\beta}\delta^+)[2\tilde{\alpha} + \sigma^2]\delta^+.$$

Proof: Note that $(1+x)^{-1} \le e^{-\gamma^- x}$ for all $x \in [0, 2\Delta_{max}\tilde{\beta}]$. From Lemma 6.1, we have

$$|x_{n+1}|^2 \le \beta_n |x_n|^2 + \kappa_n + \xi_{n+1} + \tilde{\xi}_{n+1}$$

where $\kappa_n := \Delta_n[2\tilde{\alpha} + \sigma^2]$. Defining

$$\gamma_j = \left(\Pi_{l=0}^{j-1} \beta_l^{-1}\right)$$

we obtain

$$\mathbb{E}\big(\gamma_{N_{j+1}} |x_{N_j+1}|^2 | \mathcal{F}_{N_j}\big) \le \gamma_{N_j} |x_{N_j}|^2 + \mathbb{E}\big(\sum_{l=N_j}^{N_{j+1}-1} \gamma_{l+1}\kappa_l | \mathcal{F}_{N_j}\big).$$

Now

$$\sum_{l=N_j}^{N_{j+1}-1} \Delta_j \le \delta + \Delta_{max} = \delta^+ \qquad \text{and} \qquad \sum_{l=N_j}^{N_{j+1}-1} \Delta_j \ge \delta = \delta^-. \tag{6.2}$$

Straightforward calculations show that

$$\gamma_{N_{j+1}} \ge \exp(2\tilde{\beta}\gamma^- \delta)\gamma_{N_j}$$

and

$$\gamma_{l+1} \le \exp(2\tilde{\beta}\delta^+)\gamma_{N_j}.$$

Hence

$$\mathbb{E}\Big\{|x_{N_j+1}|^2|\mathcal{F}_{N_j}\Big\} \le \exp(-2\gamma^-\tilde{\beta}\delta^-)|x_{N_j}|^2 + \exp(2\tilde{\beta}\delta^+)\mathbb{E}\Big\{\sum_{l=N_j}^{N_{j+1}-1} \kappa_l|\mathcal{F}_{N_j}\Big\}$$

and for the required result we need to bound the last term. By (6.2) we have

$$\mathbb{E}\Big(\sum_{l=N_j}^{N_{j+1}-1} \Delta_l|\mathcal{F}_{N_j}\Big) \le \delta + \Delta_{max} = \delta^+$$

and we obtain

$$\mathbb{E}\Big(\sum_{l=N_j}^{N_{j+1}-1} \kappa_l|\mathcal{F}_{N_j}\Big) \le [2\tilde{\alpha} + \sigma^2]\delta^+.$$

This gives the desired bound. $\square$

We now proceed to prove the ergodicity and moment bound. We prove geometric ergodicity of the Markov chain $\{y_j, l_j\}$ by using the approach highlighted in [23]. In particular we use a slight modification of Theorem 2.5 in [21]. Inspection of the proof in the Appendix of that paper shows that, provided an invariant probability measure exists, and this follows from Lemma 6.4, the set C in the minorization condition need not be compact: it simply needs to be a set to which return times have exponentially decaying tails.

Let

$$P(y, l, A) = \mathbb{P}((y_1, l_1) \in A|(y_0, l_0) = (y, l))$$

where

$$A \in \mathcal{B}(\mathbb{R}^m) \otimes \mathcal{B}(\mathbb{Z}^+), (y, l) \in \mathbb{R}^m \times \mathbb{Z}^+.$$

We write $A = (A_y, A_l)$ with $A_y \in \mathcal{B}(\mathbb{R}^m)$ and $A_l \in \mathcal{B}(\mathbb{Z}^+)$.

The minorization condition that we use, generalizing that in Lemma 2.5 of [21], is now proved:[1]

Lemma 6.5. *Let C be compact. For $\delta > 5\Delta_{max}$ there is $\zeta > 0$, $\overline{y} \in \mathbb{R}^m$ and $\varepsilon > 0$ such that*

$$P(y, l, A) \ge \zeta\nu(A) \quad \forall A \in \mathcal{B}(\mathbb{R}^m) \times \mathcal{B}(\mathbb{Z}^+),\ (y, l) \in C \times \mathbb{Z}^+,$$

where

$$\nu(A) = Leb\Big\{B(\overline{y}, \varepsilon) \cap A_y\Big\} \cdot \mathbf{1}\Big\{k^\star(\overline{y}) \in A_l\Big\}.$$

[1] Note that although C is compact in the following, $C \times \mathbb{Z}^+$ is not.

Proof: Let $M = N_1(2\Delta_{max})$ and $N = N_1(\delta)$. Recall the definition (5.16) of $\bar{y}$. Since $t_M \leq 3\Delta_{max}$ almost surely, setting $r^2 = R^2 + B^2$ Corollary 6.3 implies we can choose positive B and R sufficiently large so that

$$\mathbb{P}\Big\{ \sup_{0\leq n\leq M} |x_n|^2 \leq r^2 \Big\} \geq \frac{1}{2}$$

and $\overline{y} \in B(0,r)$, and $C \subseteq B(0,R)$. Label this event, with probability in excess of $\frac{1}{2}$, by E_1. If E_1 occurs then there exists $l \in \{0,\ldots,M\}$ such that $k_l \leq k^*(B(0,r))$. This follows by contradiction, since otherwise $\Delta_j = 2^{j+1}\Delta_{-1}$ for $j \in \{0,\ldots,M\}$ and

$$t_M = \sum_{j=0}^{M-1} \Delta_j \leq \Delta_{max} \sum_{j=0}^{M-1} 2^{j-M} \leq \Delta_{max} \sum_{k=1}^{\infty} 2^{-k} = \Delta_{max}.$$

However $t_M \geq 2\Delta_{max}$, a contradiction. Once $k_j \leq k^*(B(0,r))$ it follows that $k_n \leq k^*(B(0,r))$ for $n \in \{l,\ldots,M\}$ as a consequence of the step-size selection mechanism.

Assume that E_1 has occurred. By choice of ϵ sufficiently small, $B(\overline{y},\epsilon) \subseteq B(0,r)$. We now choose the η_j for $j \in \{M,\ldots,N-1\}$ to ensure the event E_2 namely that

$$x_j \in B(\overline{y},\varepsilon), \quad M+1 \leq j \leq N.$$

It is possible to ensure that the event has probability $p_1 > 0$, uniformly for $X \in C$ and $k_0 \in \mathbb{Z}^+$. The fact that $x_M \in B(0,r)$ gives uniformity in $X \in C$. We prove an upper bound on the number of steps after M to get probability independent of $k_{-1} \in \mathbb{Z}^+$. To do this notice that $k_n \leq k^*(B(0,r))$ now for $n \in \{j,\ldots,N\}$, again as a consequence of the step-size mechanism. In fact $k_N = k^*(B(\overline{y},\epsilon)) = k^*(\overline{y})$. This follows because an argument analogous to that above proves that there is $l \in \{M+1,\ldots,N\}$ for which $k_n \leq k^*(B(\overline{y},\epsilon)) = k^*(\overline{y})$ for $n \in \{l,\ldots,N\}$. Now $k^*(B(\overline{y},\epsilon)) = k^*(\overline{y})$, by continuity of f and possibly by further reduction of ϵ. Since $k_j < k^*(\overline{y}) = \mathcal{G}(\overline{y},1)$ is not possible, it follows that $k_N = k^*(\overline{y})$.

If E_1 and E_2 both occur then, for some $\gamma > 0$, the probability that $y_1 = x_{N_1(\delta)} \in A_y$ is bounded below by $\gamma Leb\{A_y \cap B(\overline{y},\varepsilon)\}$, for some $\gamma > 0$, because

$$x_N = x^\star_{N-1} + \sqrt{\Delta_{N-1}}g(x^\star_{N-1})\eta_N,$$

x_{N-1} is in a compact set and g is invertible. The fact that η_j are i.i.d Gaussian gives the required lower bound in terms of Lebesgue measure. The final result follows with $\zeta = \gamma p_1/2$. □

With this minorization condition in hand, we turn to the proof of ergodicity.

Proof of Theorem 3: The existence of an invariant measure π follows from the Foster-Lyapunov drift condition of Lemma 6.4 which gives tightness of the Krylov–Bogoljubov measures. Lemma 6.4 shows that the chain $\{y_j, l_j\}$ repeatedly returns to the set $C \times \mathbb{Z}^+$ and that the return times have exponentially decaying tails. This generalizes Assumption 2.2 in [21]. Lemma 6.5 gives a minorization condition enabling a coupling. Together these two results give Theorem 3, by applying a straightforward modification of the arguments in Appendix A of [21]. □

Proof Theorem 4: We define the stopping time N by

$$N = \inf_{n\geq 0}\{n : t_n \geq T\}$$

noting that

$$T \leq t_N \leq T + \Delta_{max}$$

Notice that (2.5) implies that

$$\sup_{0\le t\le t_N} |\overline{X}(t)|^2 \le C\left[1+\sup_{0\le k\le N}|x_k|^2+\sup_{(t-s)\in[0,\Delta_{max}],s\in[0,T]}|W(t)-W(s)|^2\right].$$

Here we have used the fact that, by Lemma 6.1,

$$|x_k^\star|^2 \le |x_k|^2 + 2\Delta_{max}\tilde{\alpha}.$$

From this relationship between the supremum of moments of $\overline{X}(t)$ and $X(t)$, and from the properties of increments of Brownian motion, it follows that, to prove Theorem 4, it suffices to bound

$$\mathbb{E}\exp(\lambda \sup_{0\le k\le N}|x_k|^2)$$

for some $\lambda > 0$. However this follows from the fact that $t_N \le T + \Delta_{max}$ and Corollary 6.3.

7. Numerical Experiments: Pathwise Convergence

We now provide some numerical experiments to complement the analysis of the previous sections. We begin, in this section, by demonstrating the importance of Assumption 3.2 in ensuring pathwise convergence. In the next section we discuss an abstraction of the method presented and studied in detail in this paper. Section 9 then shows how this abstraction leads to a variant of the method discussed here, tailored to the study of damped-driven Hamiltonian systems. We provide numerical experiments showing the efficiency of the methods at capturing the system's invariant measure.

In the convergence analysis, we made Assumption 3.2 the second part of which was to assume that the hitting time of small neighbourhoods of the set $\Psi(0)$ is large with high probability. We now illustrate that this is not simply a technical assumption. We study the test problem

$$dy = y - y^3 + dW, \tag{7.1}$$

where W is a real valued scalar Brownian motion. The set $\Psi(0)$ comprises the points where $f(y) := y - y^3$ satisfies $f(y) = 0$ and $f'(y) = 0$, that is the points $\pm 1, 0, \pm\frac{1}{\sqrt{3}}$. Since the problem is one dimensional the hitting time to neighbourhoods of these points is not small.

For contrast we apply the algorithm to the systems in two and three dimensions found by making identical copies of the equation (7.1) in the extra dimensions with each dimension driven by an independent noise. Thus the set $\Psi(0)$ comprises the tensor product of the set $\pm 1, 0, \pm\frac{1}{\sqrt{3}}$ in the appropriate number of dimensions. Small neighbourhoods of this set do have large hitting time, with high probability. To illustrate the effect of this difference between hitting times we show, in Figure 7.1, the average time step taken at a given location in space for (the first component of) y. Notice that in one dimension the algorithm allows quite large average steps in the neighbourhood of the points $\pm 1, 0, \pm\frac{1}{\sqrt{3}}$. This does not happen in dimensions two and three because the probability that the other components of y is also near to the set $\pm 1, 0, \pm\frac{1}{\sqrt{3}}$ at the same time is very small. The effect of this large choice of time steps in one dimension is apparent in the empirical densities for (the first component of) y which are also shown in Figure 7.1; these are generated by binning two hundred paths of the SDE (7.1) over two hundred time units. It is important to realize that, although the algorithm in one dimension makes a very poor approximation of the empirical density, this occurs only because of a relatively small number of poorly chosen time-steps. Figure 7.2 shows a histogram of the timesteps (k_n values) taken in one, two and three dimensions. The plots are nearly identical, except that in one dimension the algorithm allows the method to take a small number of larger steps with $k_n = 4$.

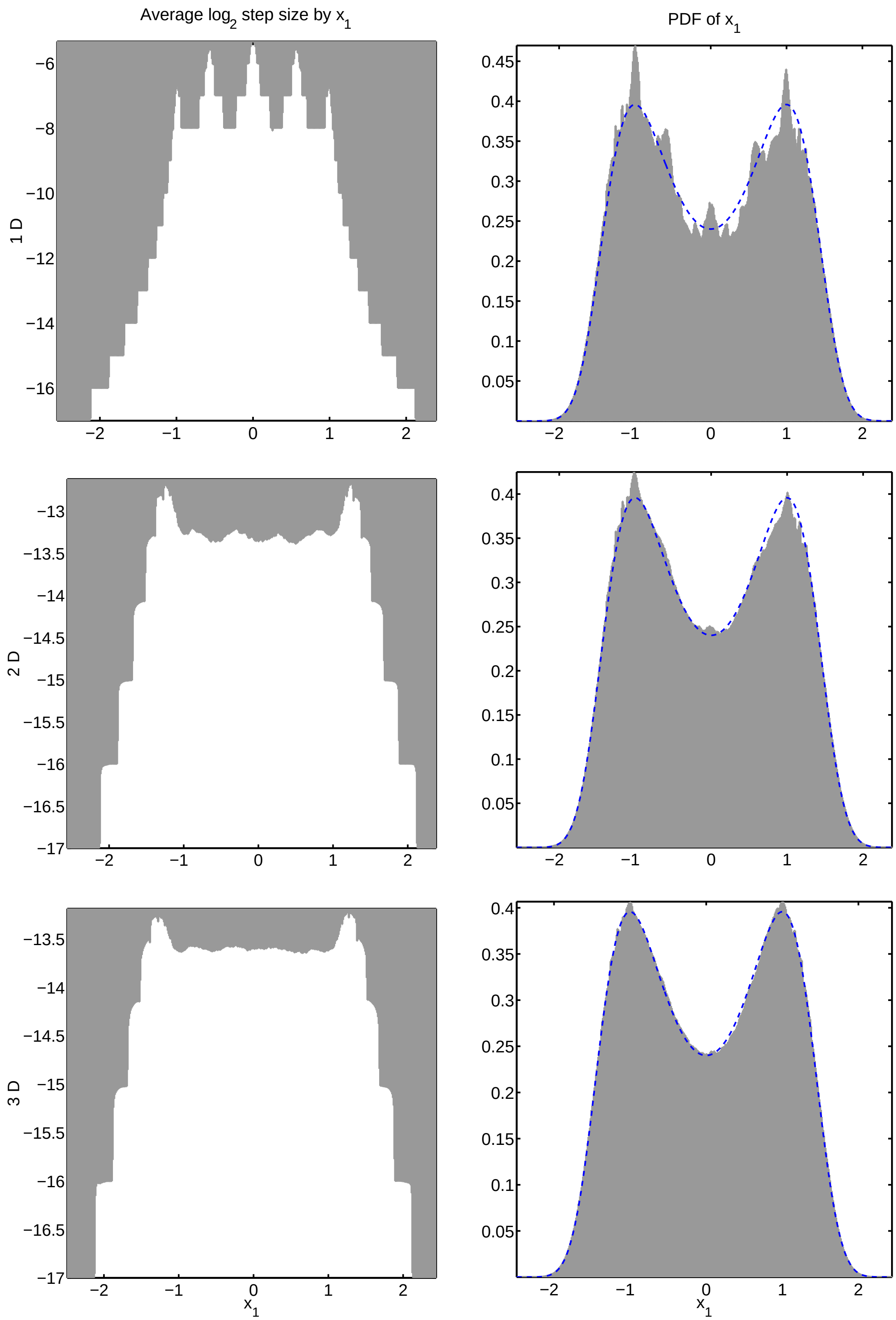

FIGURE 7.1. Effect of the "bad set" $\Psi(\epsilon)$ in different dimensions. On the right, the numerically obtained density and true analytic density (dashed line). On the left, The average *log* of the step size taken verses the spatial position of x_1.

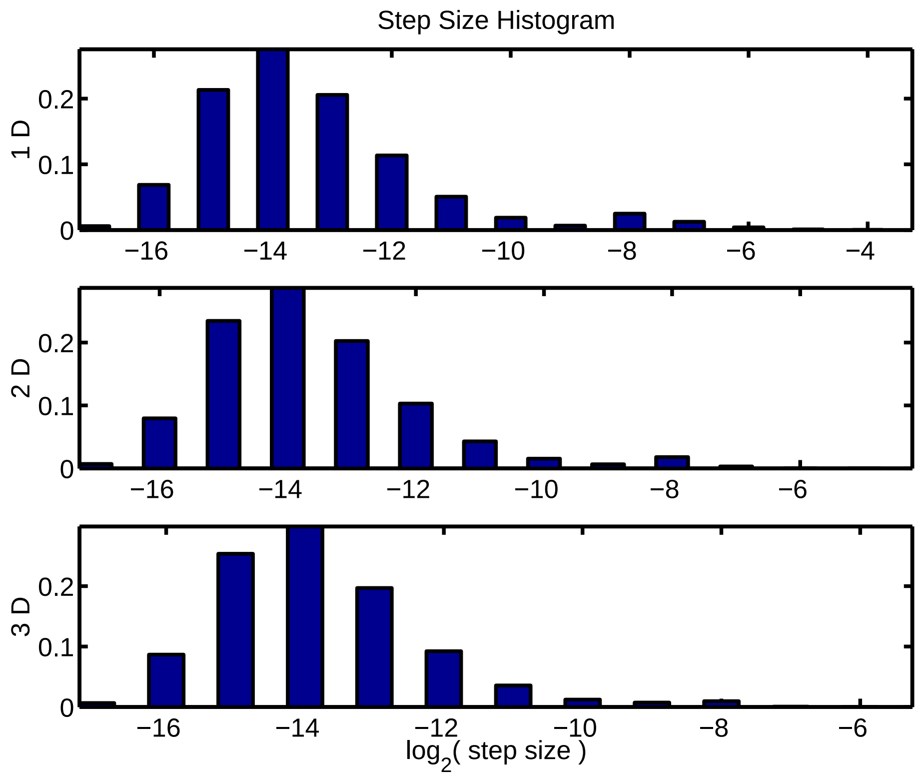

FIGURE 7.2. Gradient Problem – Timestep Historgram

8. Generalizations of the Method

The method given in (2.1) can be seen as a simple instance of a general class of methods based on comparing, with some error metric, one time step given by two methods of the form

$$
\begin{aligned}
x_{n+1} &= x_n + F(x_n, \Delta_{n+1})\Delta_{n+1} + G(x_n, \Delta_{n+1})\sqrt{\Delta}_{n+1}\eta_n \\
\overline{x}_{n+1} &= x_n + \overline{F}(x_n, \Delta_{n+1})\Delta_{n+1} + \overline{G}(x_n, \Delta_{n+1})\sqrt{\Delta}_{n+1}\eta_n \ ,
\end{aligned}
\tag{8.2}
$$

where $F, \overline{F}, G, \overline{G}$ are deterministic functions. The method in (2.1) was based on comparing the pair of explicit methods given by

$$
\begin{aligned}
x_{n+1} &= x_n^\star + \sqrt{\Delta}_{n+1} g(x_n)\eta_n \\
\overline{x}_{n+1} &= \hat{x}_n + \sqrt{\Delta}_{n+1} g(x_n^\star)\eta_n
\end{aligned}
\tag{8.3}
$$

where

$$
\begin{aligned}
x_n^\star =& x_n + \Delta_{n+1} f(x_n), \\
\hat{x}_n =& x_n + \frac{1}{2}\Delta_{n+1}[f(x_n) + f(x_n^\star)].
\end{aligned}
$$

In (2.1), closeness was measured by the difference, divided by the step size, between the conditional expectations of one time step of the two different methods; this gives

$$
\frac{2}{\Delta_{n+1}}|\mathbb{E}x_{n+1} - \mathbb{E}\overline{x}_{n+1}| = 2|F(x_n, \Delta_{n+1}) - \bar{F}(x_n, \Delta_{n+1})| = |f(x_n) - f(x_n^\star)| \ . \tag{8.4}
$$

From this point of view, it is clear that the method discussed thus far is one of a large family of methods. Depending on the setting, one might want to compare methods other that the simple Euler methods used thus far. Also one can consider different error measures. In the next section, we study a damped-driven Hamiltonian problem and use ideas from symplectic integration to design an appropriate method. In the discussion at the end of the article, we return to the question of different error measures.

9. Numerical Experiments: Long Time Simulations

In this section, we demonstrate that the ideas established for the rather specific adaptive scheme studied, and for the particular hypotheses on the drift and diffusion, extend to a wider class of SDEs and adaptive methods.

As a test problem we consider the Langevin equation

$$
\begin{aligned}
dq =& p\, dt \\
dp =& -\, \big[\delta(q)p + \Phi'(q)\big]\, dt + g(q)dW\ .
\end{aligned}
\tag{9.5}
$$

where $2\delta(q) = g^2(q)$,

$$\Phi(q) = \frac{1}{4}(1-q^2)^2 \quad \text{and} \quad g(q) = \frac{4(5q^2+1)}{5(q^2+1)}.$$

The preceding theory does not apply to this system since it is not uniformly elliptic; furthermore it fails to satisfy (1.2). However it does satisfy a Foster-Lyapunov drift condition and since it is hypoelliptic the equation itself can be proven geometrically ergodic [21]. In [21], it was shown that the implicit Euler scheme was ergodic when applied to (9.5), and a similar analysis would apply to a variety of implicit methods. Since the adaptive schemes we study in this section enforce closeness to such implicit methods, we believe that analysis similar to that in the previous section will extend to this Langevin equation and to the adaptive numerical methods studied here.

We will compare two different methods based on different choices of the stepping method. The first is the Euler based scheme given in (2.1). The second is the following scheme:

$$
\begin{aligned}
q_{n+1} =& q_n^\star \\
p_{n+1} =& p_n^\star + g(q_n^\star)\sqrt{\Delta_n}\eta_{n+1} \\
\overline{q}_{n+1} =& q_n + \Big(\frac{p_n + p_n^\star}{2}\Big)\Delta_n \\
\overline{p}_{n+1} =& p_n - \Phi'\Big(\frac{q_n + q_n^\star}{2}\Big)\Delta_n - \delta\Big(\frac{q_n + q_n^\star}{2}\Big)\Big(\frac{p_n + p_n^\star}{2}\Big)\Delta_n + g\Big(\frac{q_n + q_n^\star}{2}\Big)\sqrt{\Delta_n}\eta_{n+1}
\end{aligned}
\tag{9.6}
$$

where

$$
\begin{aligned}
q_n^\star =& q_n + p_n\Delta_n, \\
p_n^\star =& p_n - \Phi'(q_n^\star)\Delta_n - \delta(q_n^\star)p_n\Delta_n.
\end{aligned}
$$

Once again we will use comparisons between the two updates (with and without bars) to control the error. As before, we control on the difference in the expected step. The point of the particular form used here is that, in the absence of noise and damping, the adaptation constrains the scheme to take steps which are close to those of the symplectic midpoint scheme, known to be advantageous for Hamiltonian problems; if the noise and damping are small, we expect the Hamiltonian structure to be important.

In both the Euler and Symplectic case, the stepping methods take the form

$$
\begin{aligned}
q_{n+1} &= q_n + f_n^{(1)}\Delta_n \\
p_{n+1} &= p_n + f_n^{(2)}\Delta_n + g_n\sqrt{\Delta_n}\eta_{n+1} \\
\overline{q}_{n+1} &= \overline{q}_n + \overline{f}_n^{(1)}\Delta_n \\
\overline{p}_{n+1} &= \overline{p}_n + \overline{f}_n^{(2)}\Delta_n + \overline{g}_n\sqrt{\Delta_n}\eta_{n+1}
\end{aligned}
\tag{9.7}
$$

where f_n and $\overline{f}_n$, g_n, $\overline{g}_n$ are adapted to $\mathcal{F}_{n-1}$. In this notation the metric becomes,

$$
\left[(\overline{f}_n^{(1)} - f_n^{(1)})^2 + (\overline{f}_n^{(2)} - f_n^{(2)})^2\right]^{\frac{1}{2}} < \tau
$$

In the remainder of this section, we present numerical experiments with the two methods just outlined. We study the qualitative approximation of the invariant measure, we quantify this approximation and measure its efficiency, and we study the behaviour of time-steps generated.

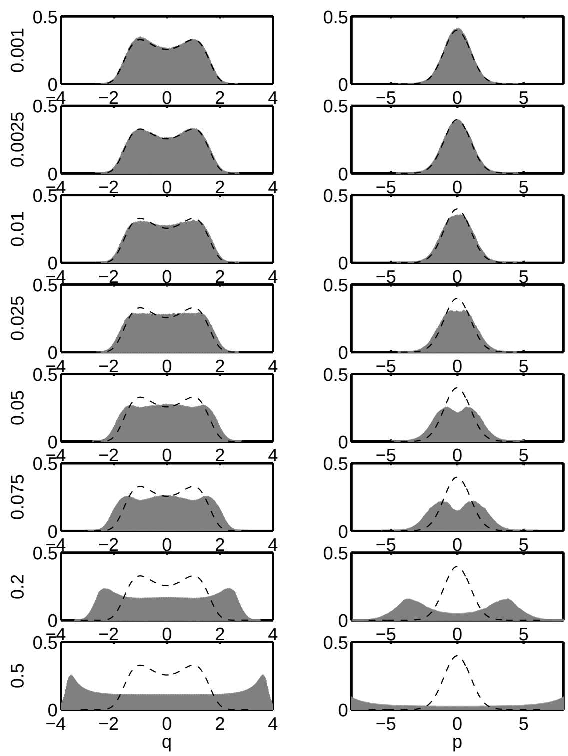

FIGURE 9.3. Distribution of q, p with different τ for Euler method The value of tolerance τ is on the left of each figure.

Figure 9 plots the numerically obtained time average of the position p and momentum q for various values of the tolerance τ. The doted lines are the true invariant measure of the underling SDE which can be computed analytically in this particular case. Notice that the method appears stable for all values of τ, in contrast to the forward Euler method which blows up when applied to this equation. Though apparently stable, the results are far from the true distribution for large τ. Figure 9 gives the analogous plots for the adaptive symplectic method given in (9.7). Notice that these methods seem to do a much better job of reproducing the invariant measure faithfully at large τ.

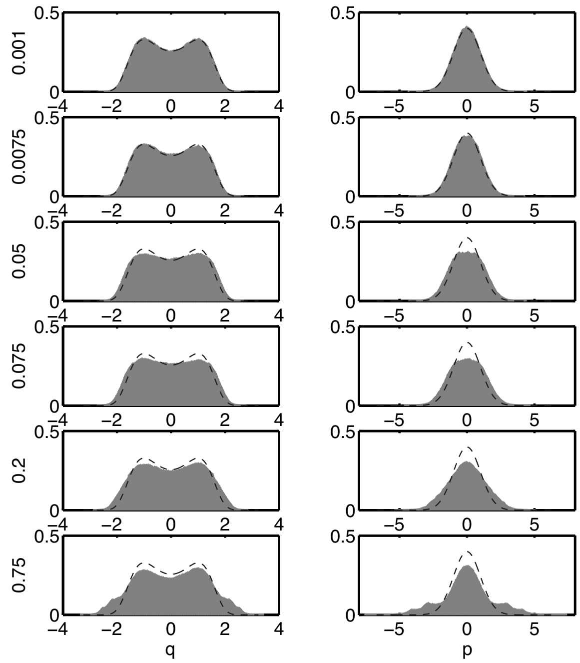

FIGURE 9.4. Distribution of q, p with different τ for the symplectic method. The value of tolerance τ is on the left of each figure.

It is also important to study accuracy per unit of computational effort. Figure 9.5 gives plots of the error in the total variation norm (the L^1 distance between the numerically computed time averages and the exact analytic answer verses the τ used and versus the steps per unit of time; the latter provides a measure of unit cost. The top plots are for the momentum q and the bottom for the position p. The plots on the right also include two fixed step methods, one using simple the forward Euler scheme and the second using the first of the symplectic schemes. The fixed-step Euler schemes blows up for steps larger than those given. We make the following observations on the basis of this experiment:

- The fixed-step symplectic method is the most efficient at small time-steps;
- The adaptive symplectic method is considerably more efficient than the adaptive and fixed-step Euler methods;
- The adaptive symplectic method is the most robust method, providing reasonable approximations to the invariant density for a wide range of τ.

Note that the adaptive methods have not been optimized and with careful attention might well beat the fixed-step methods, both as measured by accuracy per unit cost, as well as by robustness. Further study of this issue is required.

10. Conclusions

This paper proposes a simple adaptive strategy for SDEs which is designed to enforce geometric ergodicity, when it is present in the equation to be approximated; without adaptation methods such as explicit Euler may destroy ergodicity. As well as proving ergodicity, we also prove some exponential moment bounds on the numerical solution, again mimicking those for the SDE itself. Furthermore, we prove finite-time convergence of the numerical method; this is non-trivial because we do not assume (and it is not true in general) that the time-step sequence tends to zero with user input tolerance. It would be of interest to transfer this finite time convergence to a result concerning convergence of the invariant measures, something which is known for fixed time-step schemes [32, 33, 34].

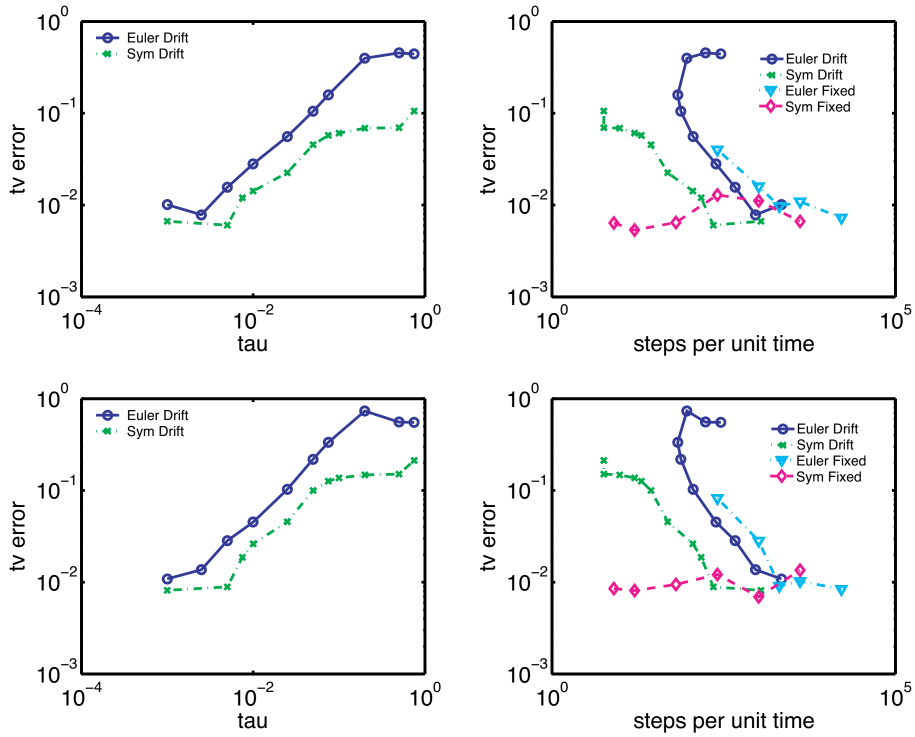

FIGURE 9.5. Total variation error verses τ for position (top left) and the momentum (bottom left). Total variation error verses τ for steps per unit time (top right) and the momentum (bottom right).

As discussed in section 8, the scheme we study in detail here is prototypical of more advanced schemes comparing two more sophisticated methods and controlling both on drift and diffusion. Here we have mainly used simple forward Euler methods and controlled only on the drift: our error measure is based on the conditional means. The split-step approach we take allows for additional terms to be added to the error measure, to ensure that the diffusion step is also controlled. The general idea is to enforce the closeness of one step by two different methods. One has freedom in the choice of the methods and the measure of closeness. We now briefly mention to other error measure which make this idea specific.

For simplicity let us assume work in one dimension though the ideas generalize directly to higher dimensions. The simple error control given in (8.4) controls only the difference in the expectation of one step of the two methods. However one can also use measure which ensure the closeness of the entire distribution of one time step of the two methods. Given $x_n = \bar{x}_n$ and Δ_{n+1}, one step of a method of the form (8.2) is Gaussian. Hence it is reasonable to require that the standard deviations are close to each other. The error criteria would then be

$$\frac{1}{\Delta_{n+1}}|\mathbb{E}x_{n+1} - \mathbb{E}\overline{x}_{n+1}| + \frac{1}{\sqrt{\Delta_{n+1}}}|\mathrm{StdDev}(x_{n+1}) - \mathrm{StdDev}(\bar{x}_{n+1})| =$$
$$|F(x_n, \Delta_{n+1}) - \bar{F}(x_n, \Delta_{n+1})| + \Big||G(x_n, \Delta_{n+1})| - |\bar{G}(x_n, \Delta_{n+1})|\Big| < \tau$$

In some ways, comparing the mean and standard deviations is rather arbitrary. A more rational choice might be to ensure the closeness of the total variation distance of the densities after one time step of the two methods. A simple way to do this is to compare the relative entropy of the two distributions. Since the distributions are Gaussian this can be done explicitly. One finds that the criteria based on controlling relative entropy per unit step is

$$\frac{\big(F(x_n,\Delta_{n+1})-\bar F(x_n,\Delta_{n+1})\big)^2}{G(x_n,\Delta_{n+1})^2}+\Big(\frac{\bar G(x_n,\Delta_{n+1})^2}{G(x_n,\Delta_{n+1})^2}-1\Big)-\log\frac{\bar G(x_n,\Delta_{n+1})^2}{G(x_n,\Delta_{n+1})^2}<2\tau\ .$$

It is interesting to note that this measure correctly captures the fact that one should measure the error in the drift on the scale of the variance. In other words if the variance is large, one does not need to be as accurate in calculating the drift as it will be washed out by the noise anyway. Since the above measure is expensive to calculate one can use that fact that $\frac{\bar G}{G}-1$ is small to obtain the asymptotically equivalent criteria

$$\frac{\big(F(x_n,\Delta_{n+1})-\bar F(x_n,\Delta_{n+1})\big)^2}{G(x_n,\Delta_{n+1})^2}+\frac{1}{2}\Big(\frac{\bar G(x_n,\Delta_{n+1})^2}{G(x_n,\Delta_{n+1})^2}-1\Big)^2<2\tau\ .$$

11. Acknowledgments

The authors thank George Papanicolaou for useful discussions concerning this work. JCM thanks the NSF for its support (grants DMS-9971087 and DMS-9729992); he also thanks the Institute for Advanced Study, Princeton, for its support and hospitality during the academic year 2002-2003. AMS thanks the EPSRC for financial support. We also wish to thank a careful and conscientious referee who noticed a substantive error in an earlier draft of this paper, and persisted until we understood his/her point.

References

[1] M.A. Aves, D.F. Griffiths and D.J. Higham, *Does error control suppress spuriosity?* To appear in SIAM J. Num. Anal.

[2] G. Cassella and C.P. Robert, *Monte Carlo Statistical Methods*, Springer Texts in Statistics, New York, 2002.

[3] K. Dekker and J.G Verwer, *Stability of Runge-Kutta Methods for Stiff Nonlinear Differential Equations.* North-Holland, Amsterdam, 1984.

[4] R.M. Dudley, *Real Analysis and Probability.* Cambridge University Press, Cambridge, 2002.

[5] J.G. Gaines and T.J. Lyons, *Variable stepsize control in the numerical solution of stochastic differential equations*, SIAM J. Appl. Math., **57**(1997),1455–1484.

[6] D.F. Griffiths, *The dynamics of some linear multistep methods with step-size control.* Appears in "Numerical Analysis 1987" Eds: Griffiths, D.F. and Watson, G.A., Longman (1988), 115–134.

[7] J.K. Hale, *Asymptotic Behaviour of Dissipative Systems.* AMS Mathematical Surveys and Monographs 25, Rhode Island, 1988.

[8] N.R. Hansen, *Geometric ergodicity of discrete time approximations to multivariante diffusions.* Bernoulli, submitted, 2002.

[9] R. Z. Has'minskii. *Stochastic Stability of Differential Equations.* Sijthoff and Noordhoff, 1980.

[10] D.J. Higham and A.M. Stuart *Analysis of the dynamics of error control via a piecewise continuous residual.* BIT **38**(1998), 44–57.

[11] D. J. Higham, X. Mao, and A. M. Stuart. Strong convergence of Euler-type methods for nonlinear stochastic differential equations. *SIAM J. Numer. Anal.*, 40:1041–1063, 2002.

[12] N. Hofmann, T. Müller-Gronbach and K. Ritter *Optimal approximation of SDEs by adaptive step-size control.* Math. Comp. **69**(2000), 1017–1034.

[13] N. Hofmann, T. Müller-Gronbach and K. Ritter *The optimal discretization of SDEs.* J. of Complexity. **17**(2001), 117–153.

[14] N. Hofmann, T. Müller-Gronbach and K. Ritter *Optimal uniform approximation of systems of SDEs.* Ann. Appl. Prob. **12**(2002), 664–690.

[15] P.E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations.* Springer-Verlag, New York, 1991.

[16] H. Lamba. Dynamical systems and adaptive time-stepping in ODE solvers. *BIT*, 40:314–335, 2000.

[17] H. Lamba and A.M. Stuart *Convergence results for the MATLAB ode23 Routine.* BIT **38**(1998), 751–780.

[18] H. Lamba, J.C. Mattingly and A.M. Stuart *An adaptive Euler-Maruyama scheme for SDEs: Part I, Convergence.*
[19] X. Mao, *Stochastic Differential Equations and Applications.* Horwood, Chichester, 1997.
[20] The MathWorks, Inc., *MATLAB User's Guide.* Natick, Massachusetts, 1992.
[21] J.C. Mattingly, A.M. Stuart and D.J. Higham *Ergodicity for SDEs and approximations: locally Lipschitz vector fields and degenerate noise.* Stoch. Proc. and Applics. **101**(2002), 185–232.
[22] J.C. Mattingly, *A numerical study of adaptive methods for SDEs.* In preparation.
[23] S. Meyn and R.L. Tweedie, *Stochastic Stability of Markov Chains.* Springer-Verlag, New York, 1992.
[24] T. Müller-Gronbach. *Strong approximation of systems of stochastic differential equations.*
[25] G.O. Roberts and R.L. Tweedie *Exponential convergence of Langevin diffusions and their discrete approximations*, Bernoulli **2**(1996), 341–363.
[26] L.C.G. Rogers and D.Williams *Diffusions, Markov processes and Martingales, Volumes 1 and 2.* Cambridge University Press, reprinted second edition, 2000.
[27] J.-M. Sanz-Serna, *Numerical ordinary differential equations versus dynamical systems.* In D.S. Broomhead and A. Iserles, Editors, "The Dynamics of Numerics and the Numerics of Dynamics", Clarendon Press, Oxford, 1992.
[28] O. Stramer and R.L. Tweedie, *Langevin-type models I: diffusions with given stationary distributions, and their discretizations*, Methodology & Computing in Applied Probability, **1**(1999), 283-306.
[29] A.M. Stuart. Probabilistic and deterministic convergence proofs for software for initial value problems. *Numerical Algorithms*, 14:227–260, 1997.
[30] A.M. Stuart and A.R. Humphries, *Dynamical Systems and Numerical Analysis.* Cambridge University Press, 1996.
[31] A.M. Stuart and A.R. Humphries, *The essential stability of local error control for dynamical systems.* SIAM J. Num. Anal. **32**(1995), 1940–1971.
[32] D. Talay, *Second-order discretization schemes for stochastic differential systems for the computation of the invariant law.* Stochastics and Stochastics Reports **29**(1990), 13–36.
[33] D. Talay, *Approximation of the invariant probability measure of stochastic Hamiltonian dissipative systems with non globally Lipschitz co-efficients.* Appears in "Progress in Stochastic Structural Dynamics", Volume 152, 1999, Editors R. Bouc and C. Soize. Publication du L.M.A.-CNRS.
[34] D. Talay, *Stochastic Hamiltonian dissipative systems with non globally Lipschitz co-efficients: exponential convergence to the invariant measure and discretization by the implicit Euler scheme.* Markov Proc. Rel. Fields., **8**(2002), 163–198.
[35] R. Temam, *Infinite Dimensional Dynamical Systems in Mechanics and Physics. Springer, New York, 1989.*

Appendix A. Two Exponential Martingale Estimates in Discrete Time

Let $\{\mathcal{F}_n,\ n \geq 0\}$ be a filtration. Let η_k be a sequence of random variables with η_k adapted to $\mathcal{F}_k$ and such that η_{k+1} conditioned on $\mathcal{F}_k$ is normal with mean zero and variance $\sigma_k^2 = \mathbb{E}[\eta_{k+1}^2|\mathcal{F}_k] < \infty$. We define the following processes:

$$M_n = \sum_{k=1}^{n} \eta_k \qquad \tilde{M}_n = \sum_{k=1}^{n} \eta_k^2 - \sigma_{k-1}^2$$

$$\langle M \rangle_n = \sum_{k=0}^{n-1} \sigma_k^2 \qquad \langle \tilde{M} \rangle_n = \sum_{k=0}^{n-1} 2\sigma_k^4$$

As the notation suggests, $\langle M \rangle_n$ and $\langle \tilde{M} \rangle_n$ are the quadratic variation processes in that $M_n^2 - \langle M \rangle_n$ and $\tilde{M}_n^2 - \langle \tilde{M} \rangle_n$ are local martingales with respect to $\mathcal{F}_n$.

Lemma A.1. *Let $\alpha > 0$ and $\beta > 0$, then the following estimate holds:*

$$\mathbb{P}\Big(\sup_k \big(M_k - \frac{\alpha}{2} \langle M \rangle_k \big) \geq \beta \Big) \leq e^{-\alpha\beta}$$

If in addition $\sigma_k^2 \le \sigma_*^2 \in \mathbb{R}$ for all $k \in \mathbb{N}$ almost surely, then

$$\mathbb{P}\Big(\sup_k \big(\tilde{M}_k - \frac{\alpha}{2}\langle \tilde{M}\rangle_k\big) \ge \beta\Big) \le e^{-\frac{\beta}{\lambda^2}}$$

where $\lambda^2 = 2\sigma_*^2 + 1/\alpha$.

Proof of Lemma A.1. We begin with the first estimate. Define $N_n = \exp(\alpha M_n - \frac{\alpha^2}{2}\langle M\rangle_n)$ and observe that $N_n = \mathbb{E}\{N_{n+1}|\mathcal{F}_n\}$. This in turn implies that $\mathbb{E}|N_n| = \mathbb{E}N_n = N_0 = 1 < \infty$. Combining these facts we see that N_n is a Martingale. Hence, the Doob-Kolmogorov Martingale inequality [26] implies

$$\mathbb{P}\big(\sup_n N_n > c\big) \le \frac{\mathbb{E}N_0}{c} = \frac{1}{c}\ .$$

Since $\mathbb{P}\Big(\sup_n (M_n - \frac{\alpha}{2}\langle M\rangle_n) \ge \beta\Big) = \mathbb{P}\big(\sup_n N_n > e^{\alpha\beta}\big)$, the proof is complete.

The second estimate is obtained in the same way after some preliminary calculations. We define $\phi(x) = \frac{1}{2}\ln(1-2x)$ and $\psi(x,b) = -x - bx^2$. Observe that $c\psi(x,b) = \psi(cx, b/c)$ and $\phi(x) \ge \psi(x,b)$ if $x \in \big[0, \frac{1}{2}\big(\frac{b-1}{b}\big)\big]$ and $b > 1$. Now

$$\mathbb{P}\left(\sup_n (\tilde{M}_n - \frac{\alpha}{2}\langle \tilde{M}\rangle_n) \ge \beta\right) = \mathbb{P}\left(\sup_n \sum_{k=1}^n \frac{\eta_k^2}{\lambda^2} + \frac{1}{\lambda^2}\psi(\sigma_{k-1}^2, \alpha) \ge \frac{\beta}{\lambda^2}\right)\ .$$

Setting $\lambda^2 = 2\sigma_*^2 + \frac{1}{\alpha}$, we have that $\frac{1}{\lambda^2}\psi(\sigma_k^2, \alpha) = \psi(\frac{\sigma_k^2}{\lambda^2}, \lambda^2\alpha) \le \phi(\frac{\sigma_k^2}{\lambda^2})$ for all $k \ge 0$ since $\sigma_k^2 \le \sigma_*^2$ and $\lambda^2\alpha > 1$. Defining

$$\tilde{N}_n = \exp\left(\sum_{k=1}^n \frac{\eta_k^2}{\lambda^2} + \phi(\frac{\sigma_{k-1}^2}{\lambda^2})\right)\ ,$$

we have

$$\mathbb{P}\left(\sup_n (\tilde{M}_n - \frac{\alpha}{2}\langle \tilde{M}\rangle_n) \ge \beta\right) \le \mathbb{P}\left(\sup_n \tilde{N}_n \ge e^{\frac{\beta}{\lambda^2}}\right)\ .$$

Now recall that if ξ is a unit Gaussian random variable then $\mathbb{E}\exp(c\xi^2) = 1/\sqrt{1-2c}$ for $c \in (-\frac{1}{2}, \frac{1}{2})$. By construction $\frac{\eta_k}{\lambda}$, conditioned on $\mathcal{F}_{k-1}$, is a Gaussian random variable with variance less then $\frac{1}{2}$. Hence

$$\mathbb{E}\Big(\exp\big(\frac{\eta_k^2}{\lambda^2}\big)\big|\mathcal{F}_{k-1}\Big) = \exp\Big(-\phi\big(\frac{\sigma_{k-1}^2}{\lambda^2}\big)\Big)\ .$$

Using this one sees that $\mathbb{E}\big\{\tilde{N}_{n+1}|\mathcal{F}_n\big\} = \tilde{N}_n$ and $\mathbb{E}|\tilde{N}_n| = 1 < \infty$, hence $\tilde{N}_n$ is a Martingale. By the same argument as before using the Doob-Kolmogorov Martingale inequality, we obtain the quoted result. $\square$