Path-dependent equilibria induced by genuinely sticky expectations

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Abstract

We investigate a simple macroeconomic model where rational inflation expectations is replaced by a boundedly rational, sticky, response to changes in the actual inflation rate. Our expectations rule differs from standard sticky models and incorporates truly 'stuck' behavior as opposed to delayed rationality. The model can be rigorously analyzed and we prove that the unique equilibrium of the rational expectations model is replaced by a continuum of potential equilibria. The equilibrium that exists at any given time depends, in a deterministic way, upon previous extreme states of the system. The agents are sufficiently far-removed from the rational expectations paradigm that indeterminacy issues do not arise.

The response to exogenous noise is far more subtle than in a unique equilibrium model. After sufficiently small shocks the system will indeed revert to the same equilibrium but larger shocks will change the equilibrium value (without changing the model parameters). The path to this new equilibrium may be very long with a highly unpredictable, sometimes counter-intuitive, endpoint. At certain model parameters exogenously-triggered runaway inflation can even occur.

Finally, we analyze a variant model in which the same form of sticky response is introduced into the interest rate rule instead.

Keywords: bounded rationality, rational expectations, sticky information, representative agents, hysteresis, adaptive expectations, path-dependence, sticky inflation.

1. Introduction

Modern macroeconomics has been dominated by a modeling framework in which the economy is assumed always to be at (or rapidly moving back towards) a unique and stable equilibrium. This has had profound implications both for the way in which the modelers perceive real-world events and their policy prescriptions for dealing with them.

The critiquing of equilibrium models has a long history which we shall not attempt to detail here. But many antagonists, see for example [1, 2, 3, 4], have eloquently pointed out profound issues concerning the assumed equilibrating processes and the ways in which the 'aggregation problem' was being solved. In this paper we will focus upon one pillar of the equilibrium approach which is the assumption of Rational Expectations introduced by

Muth in 1961 [5]. This posits that not only are individuals perfectly rational, optimizing, far-sighted and independent of each other but that their expectations about future uncertainties are in agreement with the model itself. Our mathematical analysis and the supporting numerics rigorously show that, when rational expectations about future inflation are replaced by an aggregated 'sticky' expectation, a simple macroeconomic model changes from a unique equilibrium system to one with an entire continuum of path-dependent equilibria. The magnitude of exogenous shocks is now crucial to understanding the model's behavior — sufficiently small shocks will be 'forgotten' but larger shocks will not.

The form of stickiness that we use is, to our knowledge, new in a macroeconomic setting and differs from the stickiness of Calvo pricing models [6] or the sticky-information of Mankiw and Reis [7] where hypothetical agents adjust to the 'correct' rational response at a fixed rate rather than instantaneously. For the case of inflation expectations we instead assume that the inflation expectations variable remains fixed/stuck until the difference between expectations and the actual inflation rate becomes too large (in either direction) in which case the inflation expectations value moves so as to keep the difference at this maximum permissable level. It is important to note that our agents are therefore not imbued with any concept of rational expectations, even with a delay as in [6, 7].

Indeed our model has more in common with approaches that were popular before the rise of rational expectations. In particular our expectations are 'backward-looking' as in Adapative Expectations and some Adaptive Learning [8] models which also generate path-dependent equilibria. However, rather than using, say, lagged inflation values with exponentially-decaying weights our inflation expectation term is very simply determined by a subset of the most recent extrema of the actual inflation rates. Our hypothesized representative agent may have extremely limited computational powers but the result is a simple model that captures important aspects of bounded rationality, inertia and anchoring, and displays realistic non-rational dynamics.

The way in which we incorporate this form of inflation expectations stickiness into our macroeconomic models will be justified further and described precisely below but, more generally, our sticky variables can only be in one of two modes. They are either currently 'stuck' at some value or they are being 'dragged' along by some other (related) variable because the maximum allowable difference between them has been reached. Each of these modes (which we shall also refer to as the 'inner' and 'outer' modes respectively) can be analyzed separately as linear systems using standard stability techniques. However the full 'hybrid' system is nonlinear and displays far richer dynamics in the presence of exogenous noise and shocks.

It must be emphasized right away that our modeling approach and analytical tools are not restricted to inflation expectations or even to macroeconomics. The form of stickiness described above is represented by a class of mathematical objects that have well-understood and very desirable properties. These have already been used to develop non-equilibrium asset-pricing models [9] that have (almost-) analytic solutions and are also being studied in various micro-economic settings.

Here we are able to prove the existence of an entire line interval of feasible equilibrium points, examine their stability, and identify some important consequences of path dependence regarding the effects of exogenous shocks and policy changes upon the state of the system. Furthermore, these effects are plausible in that they both correspond closely to observed, but potentially puzzling, economic situations and are robust enough to be observed numerically in more sophisticated variants of the model.

The level of mathematical knowledge required to follow most of the arguments is not much more than is needed to examine the existence and stability of equilibria in more traditional, fully linear, models. Another useful aspect of this simple model is that the stickiness can be smoothly 'dialed back' to zero and the unique equilibrium case is recovered. Or, to put it another way, we can rigorously show that a plausible, *boundedly rational yet fully analyzable*, change to a fully rational model significantly alters the qualitative behavior of the system in recognizable ways.

Before introducing the model and starting the analysis, it is worth stepping back to consider the effects of stickiness and friction in physical rather than economic systems. This helps develop our intuition about the nature of equilibria in such systems but the comparison also offers a high-level explanation of the difficulties faced by mainstream equilibrium-based economics both in foreseeing economic crises and and attempting to reverse their effects.

1.1. Economics, Earthquakes and Friction

In early 2009, Alan Greenspan, former Chairman of the Federal reserve, wrote the following:

"We can model the euphoria and the fear stage of the business cycle. Their parameters are quite different. ... we have never successfully modeled the transition from euphoria to fear."
— Alan Greenspan, *Financial Times*, March 27th 2009.

The implication is that Central Bank models work well 'most of the time' with suitably calibrated parameters. Occasionally the parameters suddenly change but once these are measured the model again works well in the neighborhood of a new equilibrium.

The above response to models that suddenly fail is only justified when the transitions between euphoria and fear and the (assumed) change in parameters are truly exogenously triggered. If they are due to endogenous causes then the model was never really working before the transition and it probably won't after the transition either!

There is a useful analogy with earthquakes and seismology[1]. Earthquake zones *appear* to be stable (i.e. in an equilibrium) for very long periods of time with only very brief, but violent, 'transitions'. A tectonic-plate-denying 'equilibrium seismologist' might argue that the earthquake-free equilibrium model was essentially correct except for some occasional unpredictable exogenous events (unobserved meteorite strikes!?) that didn't in any way cast doubt on the modeling assumptions.

[1]Much has been published about the fact that the frequencies of both earthquakes and financial crashes appear to have power-law distributions with some authors even drawing very close mathematical parallels between the mechanisms responsible. However here we are only using earthquakes as an aid to understanding some of the effects of friction and barriers to equilibration in real-world systems.

Of course, earthquakes are almost always endogenously generated and the analogy can be pushed further. An earthquake is a very fast shift from one (meta-)stable[2] internal configuration to another and this leads us consider the concept of 'balance-of-forces' in both physics and economics.

Ever since the time of Walras and Jevons the idea that there should be a complete and unique set of equilibrium prices that exactly balances all of the competing needs and desires of economic agents has offered a compelling view of a perfectly balanced economy with tâtonnement processes somehow achieving this outcome. But this view is based upon a comparison with physical systems that is misleading. A spring or piece of elastic subject to competing forces will achieve a unique equilibrium but this is because there is no complex internal structure capable of absorbing any of the stresses without yielding.

A more complicated physical system such as a tectonic fault line has myriad internal configurations capable of balancing the forces applied to it — up to a point. Which particular configuration exists at any given moment will depend upon the previous states of the system. And when one small part of the fault line suddenly shifts this can transfer excess stress to neighboring parts resulting in a large cascading failure/earthquake. There is a balance of forces before the earthquake and after the earthquake but not *during* the earthquake!

Modern economies are arguably the most complicated man-made constructs on the planet with an immensely intricate internal description which cannot simply be averaged away. The analogy is also useful in that the fundamental source of earthquakes is friction. Without it, continental plates would gracefully and safely glide rather than stick and grind. Frictions and stickiness are present in many forms in an economy or financial system and it should not be a surprise if they cause similar qualitative effects — even though fault lines are being consistently forced in a single direction while the changes experienced by economies are more random.

This brings us to a crucial discussion of *timescales*. In a pure equilibrium system there is no notion of *any* timescale except for ones imposed exogenously[3]. The system is completely determined by the current values of potentially a small number of variables (for example, the pressure, volume and temperature of gas in a container). If one introduces external fluctuations and/or deliberate interventions then there will of course be some delay as the system tries to settle at a new equilibrium but the conceptual picture remains fundamentally unchanged — only the most recent history is relevant as influences typically fade exponentially.

However, earthquakes provide a perfect example of how frictions and barriers to equilibration can introduce surprisingly long timescales into a system via the existence of metastable states. Studying an earthquake fault on a timescale of days or weeks or even years may easily lead to the potentially disastrous conclusion that a stable linear equilibrium-based model is appropriate — small barely detectable tremors occur frequently but they also die away

[2]Metastability in physics is when a system can stay in a particular state for an indefinite amount of time even though it is not the state of lowest energy. It occurs when there is some kind of barrier to true equilibration.

[3]There is no notion of history either. If a system is at its unique equilibrium there is no way of telling where it has been.

extremely rapidly as the system, on the surface at least, appears to return to its previous state. But, if looked at on a long-enough timescale, maybe tens of thousands of years, then it doesn't look like an equilibrium at all — rather, a series of very unpredictable violent shifts between different, temporarily feasible, internal states.

If economies feel like they are close to a unique equilibrium maybe that's just because most of the time tomorrow does indeed turn out to be a lot like yesterday! Over short timescales unique equilibrium models will indeed *appear* to work well most of the time — especially if their parameters are being frequently tuned to match the incoming real-world data they are trying to predict! But over longer timescales one should not be surprised that sudden and unexpected transitions (such as between euphoria and fear) occur and should not be too quick to ascribe them to exogenous influences or even parameter changes.

1.2. Permanence and Path-Dependence

If the presence of stickiness/frictions in economics does indeed induce a myriad of co-existing equilibria then phenomena that are not possible (or require a posteriori model adjustments) in unique equilibrium models become not just feasible but inevitable. Perhaps the most obvious of these is *permanence*, also known as remanence, where a system does not revert to its previous state after an exogenous shock is removed. It is of course a central concern of macroeconomics whether or not economies affected by, say, significant negative shocks can be expected to have permanently reduced productivity levels.

For the models studied in this paper, sufficiently small shocks (whether exogenous or applied by policy makers) will not change the equilibrium point and a standard linear stability analysis determines the rate at which the system returns to it. Larger shocks will move the equilibrium point along a line of potential equilibria in the expected direction. But even larger shocks may move the system far enough away from the equilibrium interval that the return path and ending point on the interval are very hard to predict. Furthermore, in neither of the last two cases will the system exhibit any tendency to return to its pre-shocked state — the model displays true permanence. And a related property is that the model parameters alone cannot determine which equilibrium a system is currently in without knowing important information about the prior states of the system — true path dependence (note that this does not prevent the system from being iterated once the initial conditions are *fully* specified).

1.3. Sticky Models and Indeterminacy

The most widely-used sticky models are the sticky-prices of Calvo [6] and the sticky-information of Mankiw and Reis [7]. These models are conceptually very similar to each other in that agents do not all instantaneously move to the 'correct' price or opinion but rather do so at a fixed rate and can be represented mathematically by introducing delay terms into the relevant equations. In the absence of noise the same optimal equilibrium solution will be reached as if the stickiness were absent.

Continua of possible equilibria can also occur in such models (see [10, 11] and for the special case of passive interest-rate policy see [6, 12]) and is considered an extreme form of *indeterminacy*. This is problematic within a Rational Expectations framework since it makes it (even!) harder to justify how the agents' expectations can be consistent with the model.

Our hypothetical agents are less rational than those above. They are truly stuck (not just delayed) until forced to adjust by the magnitude of the discrepancy with the actual inflation rate and they have no assumed awareness of the modeling assumptions. This means that at any moment in time the particular equilibrium being approached is determined by prior states of the system and *not* by modeling assumptions about the future.

The research into how expectations are actually formed is extensive but far from conclusive, see for example [13, 14, 15, 16, 17]. We focus on two observations that help justify studying models with our new form of stickiness. Firstly, the ideas of threshold effects and a 'harmless interval' of inflation are not new in economics [18, 19, 20, 21, 22] and are consistent with our modeling approach. Indeed there is some evidence for exactly our form of stickiness in experimental data [23, 24].

Secondly, since at least Keynes and his General Theory the idea that humans will make boundedly rational shortcuts or use 'rules of thumb' has been an important element in certain schools of macroeconomics, especially when the future is highly unpredictable. Various theories of 'Heuristics under Uncertainty' and rules for 'Satisficing' have been observed by experimental economists and formalized by theorists [25, 26]. The stickiness model we use combines aspects of inertia and anchoring together with a minimal 'expectations adjustment' procedure into an analytically tractable alternative to staggered/delayed models — one capable of additional complexity and explanatory power.

1.4. Bounded Rationality and Aggregation

As mentioned above, the standard approach to the problem of aggregating expectations is to introduce a 'Representative Agent' whose expectations are fully-informed and rational and consistent with the model itself. Here, an aggregation of *boundedly* rational agents into a similar Representative is required.

Our approach is similar in spirit to that of De Grauwe [27] although the details of our less-than-perfectly-rational agents are quite different. In [27] both the expectations terms in inflation and output gap are linear combinations of the expectations of two kinds of agent — rational 'fundamentalists' and boundedly rational 'extrapolators' — with the probability of an agent using each being dictated by discrete choice theory [28, 29]. He then showed numerically that cycles of booms-and-busts occurred with changes in the 'animal spirits' and corresponding non-Gaussian 'fat-tailed' distributions for the model variables. Discrete choice theory is the aggregating mechanism that De Grauwe uses to avoid ending up with an agent-based model where each agent has to be individually simulated.

We use the empirical evidence cited above that individual agents' expectations are often sticky and may lag behind the currently observable values before they start to move. We also posit, quite reasonably, that this gap between future expectations and current reality cannot grow too large. We then imbue our now boundedly rational Representative Agent with these same properties. This leads us in a very natural way to the model that is described fully in Section 2.1.

While this is certainly not a fully-justified aggregation procedure neither are the others mentioned above! Once one accepts that generating a realistic simulation of all the less-than-rational agents in an economy is not feasible, using boundedly rational representative agents that inherit some basic properties from their constituents is necessary to provide insights into the macro-effects of bounded rationality.

This brings us to another very important issue concerning rational expectations and indeed rationality assumptions in general. A major shortcoming in the arguments used to generate rationality-based models is that they typically give no information, either quantitative or qualitative, about what happens if the rationality assumptions are weakened or are 'not quite true'. Or to put it another way, they do not address the structural stability of the proposed rational solution in the presence of boundedly rational perturbations. It seems to be implicitly assumed by the users of such models that the best-case scenario holds — the level of irrationality in the real-world agents does not significantly change the equilibrium nature of the solution or the calculation of the equilibrium position.

The space of all possible boundedly rational perturbations is very large and very hard to study rigorously or even define. This makes the analysis of particular, plausible, boundedly-rational variants of rational models a subject of independent interest and provides a secondary justification for our non-standard representative agent.[4]

1.5. Outline of the paper

We start from a dynamic stochastic general equilibrium (DSGE) macroeconomics model, which includes aggregate demand and aggregate supply equations

$$\begin{aligned} y_t &= y_{t-1} - a(r_t - p_t) + \epsilon_t, \\ x_t &= b_1 p_t + (1 - b_1) x_{t-1} + b_2 y_t + \eta_t \end{aligned} \tag{1}$$

augmented with the rate-setting rule

$$r_t = c_1 x_t + c_2 y_t, \tag{2}$$

where y_t is the output gap (or unemployment rate, or another measure of economic activity such as gross domestic product), x_t is inflation rate, r_t is interest rate, p_t is the economic agents' aggregate expectation of future inflation rate and ϵ_t, η_t are exogenous noise terms. Time is an integer variable, $t = 1, 2, \ldots$, and the process starts from initial values x_0, y_0, p_0. All the parameters are non-negative and in addition, $b_1 < 1$. This model is close to the starting model used in [27] but simpler in that we do not include the aggregate expectation of the output gap and the correlation between the subsequent values of the interest rate. We also choose to remove the noise term from the interest rate update rule. The inclusion of such factors does not affect our most significant qualitative observations, but would complicate some aspects of the rigorous analysis that we present.

[4] In [30] a similar 'stress test' was applied to equilibrium models used in finance. It was shown that even very low levels of irrational or perversely-incentivized herding by market participants will destabilize the equilibrium (Brownian motion) solution for an asset price and replace it with 'boom-and-bust' dynamics that is only evident over long timescales.

The novelty of our modeling strategy is in how we define the relationship between the aggregate expectation of inflation p_t and the inflation rate x_t. This relationship is defined precisely in the next section where we introduce the play operator to model the economic agents' aggregate expectation of future inflation.

In Sections 2.4-2.5 we present the main stability analysis for various parameter regimes, with some details relegated to Appendices. The stability properties of the system are not as clear cut as in a truly linear system. In fact, our equations define a *piecewise linear* (PWL) system, and certain nonlinear effects come into play. In particular, in nonlinear systems an equilibrium may only be *locally* stable. This means that the equilibrium is only stable to perturbations of a certain size — ones that don't move the system outside of a 'basin of attraction' — and this phenomenon is responsible for much of the interesting dynamics in the presence of shocks of differing sizes.

In Sections 3.1-3.6 we present various numerical simulations. We are particularly interested in the transitions between equilibrium states caused by exogenous shocks, and the effects of increasing or decreasing stickiness. Where possible we compare results against the non-sticky model. Permanence is the rule not the exception and there are even parameter regimes where a large enough shock will completely destabilize an apparently stable system via a runaway inflation mechanism. We also compare the statistical output of the model against that of De Grauwe [27] at similar parameters and see the same boom-and-bust cyclicality and heavy-tailed distributions.

Then, in Section 3.7 we briefly consider a more complicated version of the model with three representative agents all with different levels of stickiness. This is primarily to demonstrate that multiple play operators can indeed be used together to simulate different representative agents within a model and that the most important qualitative features are unchanged.

Finally, in Section 3.8 we emphasize that play operators are not just a potential tool for modeling expectations by removing the stickiness from the inflation expectations and adding it into the response of the Central Bank instead. We perform a second stability analysis and obtain some interesting new effects — there is the possibility of (quasi)-periodic behavior in the absence of noise and sticky Central Bankers appear to destabilize equilibria. We conclude with a summary of the main results, some general implications for policy and modeling, and some suggestions for future work.

2. The model

2.1. Play and Stop Operators

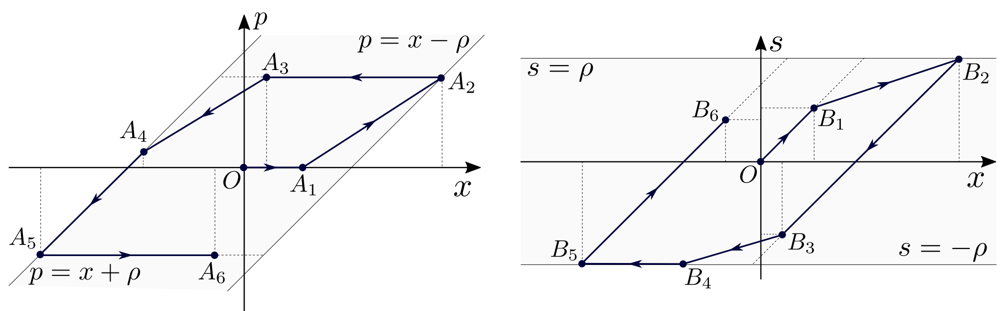
We assume the following rules that define the variations of the expectation of future inflation rate p_t with the actual inflation rate x_t at integer times t:

(i) The value of the difference $|p_t - x_t|$ never exceeds a certain bound ρ;

(ii) As long as the above restriction is satisfied, the expectation does not change, i.e. $|x_t - p_{t-1}| \leq \rho$ implies $p_t = p_{t-1}$;

(iii) If the expectation has to change, it makes the minimal increment consistent with constraint (i).

Rule (ii) introduces stickiness in the dependence of p_t on x_t, while (i) states that the expected inflation rate cannot deviate from the actual rate more than prescribed by a threshold value ρ. Hence p_t follows x_t reasonably closely but on the other hand is conservative because it remains indifferent to variations of x_t limited to a (moving) window $p - \rho \leq x \leq p + \rho$. The last rule (iii) enforces continuity of the relationship between p_t and x_t and, in this sense, can be considered as a technical modeling assumption that is mathematically convenient.



(a) (b)

Figure 1: (a) An illustration of the input-output sequence of the (a) play operator and (b) stop operator. (a) The polyline $OA_1A_2A_3A_4A_5A_6$ represents a sample input-output trajectory for the play operator. The input-output pair (x, p) is bounded to the gray strip between the two parallel lines $p = x \pm \rho$. In [24], this strip is called *band of inactivity*, the line $x = x - \rho$ is called *upward spurt* line while the line $p = x + \rho$ is called *downward spurt* line. The output p remains unchanged for a transition from (x_{t-1}, p_{t-1}) to the next point (x_t, p_t) as long as the pair (x_t, p_{t-1}) fits to the band of inactivity (for example, the transitions from $A_2 = (x_2, p_2)$ to $A_3 = (x_3, p_3)$ with $p_2 = p_3$ or from $A_5 = (x_5, p_5)$ to $A_6 = (x_6, p_6)$ with $p_5 = p_6$). If $x_t > x_{t-1}$ and the point (x_t, p_{t-1}) lies to the right of the inactivity band, then the output increases resulting in the point (x_t, p_t) to lie on the upward spurt curve (for example, the transition from $A_1 = (x_1, p_1)$ to $A_2 = (x_2, p_2)$). Similarly, if $x_t < x_{t-1}$ and the point (x_t, p_{t-1}) lies to the left of the inactivity band, then the output decreases and the point (x_t, p_t) lies on the downward spurt line (for example, the transition from $A_3 = (x_3, p_3)$ to $A_4 = (x_4, p_4)$). (b) The input-output trajectory of the dual stop operator corresponding to the trajectory of the play operator shown in panel (a). Here $s_t = x_t - p_t$; the trajectory is limited to the horizontal strip $-\rho \leq s \leq \rho$ at all times.

Rules (i)–(iii) are expressed by the formula

$$p_t = x_t + \Phi_\rho(p_{t-1} - x_t) \tag{3}$$

with the piecewise linear saturation function

$$\Phi_\rho(x) = \begin{cases} \rho & \text{if} \quad x \geq \rho, \\ x & \text{if} \quad -\rho < x < \rho, \\ -\rho & \text{if} \quad x \leq -\rho. \end{cases} \tag{4}$$

Relationship (3) is known as the *play* operator with *threshold* ρ, see Fig. 1(a). A dual relationship

$$s_t = \Phi_\rho(x_t - x_{t-1} + s_{t-1}) \tag{5}$$

between x_t and the variable

$$s_t = x_t - p_t$$

is referred to as the *stop* operator, see Fig. 1(b). In the context of our model, s_t measures the difference between the inflation rate and the expectation of the future inflation rate, hence s_t remains within the bound $|s_t| \leq \rho$ at all times. Interestingly the explicit relationship (3) has been observed in actual economic data [23, 24].

One can think of the play operator as having two modes. A 'stuck mode' where it will not respond to small changes in the input and a 'dragged mode' where the absolute difference between the input and output are at the maximum allowable and changes to the input, in the correct direction, will drag the output along with it.

Equations (3) and (5) will now be denoted by

$$p_t = \mathcal{P}_\rho[x_t], \qquad s_t = x_t - p_t = \mathcal{S}_\rho[x_t], \tag{6}$$

where $\mathcal{P}_\rho$ and $\mathcal{S}_\rho$ are the *play* and *stop* operators with threshold ρ, respectively.

2.2. A model with sticky inflation expectations

Equations (1) and (2), completed with formulas (3) and (4), form a closed model for the evolution of the aggregated variables x_t, y_t, r_t, p_t. However, the dependence of these quantities at time t upon their values at time $t-1$ is implicit. In order to implement the model, we proceed by solving equations (1)–(4) with respect to the variables x_t, y_t. As shown in Appendix A, the model can be written in the following equivalent form:

$$z_t = Az_{t-1} + s_t d + N\xi_t \tag{7}$$

where $z_t = (y_t, x_t)^\top$, $\xi_t = (\epsilon_t, \eta_t)^\top$, the superscript $^\top$ denotes transposition, the matrices A, N and the column vector d are defined by

$$A = \frac{1}{\Delta}\begin{pmatrix} 1-b_1 & a(1-b_1)(1-c_1) \\ b_2 & (1-b_1)(1+ac_2) \end{pmatrix}, \quad N = \frac{1}{\Delta}\begin{pmatrix} 1-b_1 & a(1-c_1) \\ b_2 & 1+ac_2 \end{pmatrix}, \tag{8}$$

$$d = \frac{1}{\Delta}\begin{pmatrix} a(b_1c_1 - 1) \\ -(ab_2 + b_1(1+ac_2)) \end{pmatrix}$$

with

$$\Delta = (1-b_1)(1+ac_2) + ab_2(c_1 - 1) \tag{9}$$

and $s_t = x_t - p_t$ is defined by the equation

$$s_t = \frac{1}{1+\alpha}\,\Phi_{(1+\alpha)\rho}(f_t - f_{t-1} + s_{t-1}) \tag{10}$$

with

$$\alpha = \frac{\Delta}{b_1(1+ac_2) + ab_2}, \tag{11}$$

$$f_t = \frac{\alpha}{\Delta}\big(b_2 y_{t-1} + (1-b_1)(1+ac_2)x_{t-1} + b_2\epsilon_t + (1+ac_2)\eta_t\big). \tag{12}$$

Equations (7), (10) express y_t, x_t and $s_t = x_t - p_t$ explicitly in terms of the previous values of the same variables and the exogenous noise ϵ_t, η_t. We use these equations in all the simulations that follow.

We shall refer to the variable $s_t = x_t - p_t$ as the *perception gap*. Note that (10) defines a stop operator with input f_t and threshold $(1+\alpha)\rho$, which is different from ρ (cf. (4)) and so (10) can be written as

$$s_t = \frac{1}{1+\alpha}\mathcal{S}_{(1+\alpha)\rho}[f_t]$$

using the notation (6). It is important to note that the transition to equations (7), (10) is justified under the condition that α is positive, and we assume this constraint to hold in the rest of the paper. In particular, $\alpha > 0$ whenever $c_1 > 1$ (see Section 2.5).

2.3. An entire line segment of equilibrium points

We begin the analysis of the model (7), (10) by looking at the case of no exogenous noise, i.e. we set $\xi_t = 0$ and consider the equation

$$z_t = Az_{t-1} + s_t d, \qquad z_t = (y_t, x_t)^\top \tag{13}$$

instead of (7) with s_t defined by (10), (11) and

$$f_t = \frac{\alpha}{\Delta}\big(b_2 y_{t-1} + (1-b_1)(1+ac_2)x_{t-1}\big). \tag{14}$$

This model has an entire line segment of equilibrium points which corresponds to a continuum of feasible equilibrium states of the economy as a function of the inflation expectations of economic agents. Indeed, equation (13) implies

$$z_* = s_*(\mathbb{I} - A)^{-1} d = s_* \left(\tfrac{b_1}{b_2}, \quad \tfrac{b_2 + b_1 c_2}{b_2(1-c_1)}\right)^\top \tag{15}$$

for an equilibrium point $z_* = (x_*, y_*)^\top$, where $\mathbb{I}$ is the 2×2 identity matrix. Hence one obtains a different equilibrium for each admissible value of the perception gap variable s_*, i.e. $-\rho \le s_* \le \rho$. Thus, the set of all equilibrium points, which can be denoted as $z_*(s_*)$ for different s_*, can be naturally thought of as a line segment in the phase space of the system, see Fig. 2. In particular, the value of the output gap at an equilibrium, $y_*(s_*)$ ranges over the interval $[-\rho b_1/b_2, \rho b_1/b_2]$ and the equilibrium value of the actual inflation belongs to the range

$$x_*(s_*) = s_* \frac{b_2 + b_1 c_2}{b_2(1-c_1)} \quad \text{with} \quad -\rho \le s_* \le \rho.$$

Interestingly, at least in this simple model, the range of equilibrium values of the output gap is unaffected by the controls c_1, c_2 applied by the regulator through Taylor's rule (2). However, these controls do affect the range of possible values of the equilibrium inflation rate.

Equation (15) indicates the difference between the cases $c_1 > 1$ and $c_1 < 1$. When $c_1 > 1$, the equilibrium $z_*(\rho)$ corresponding to the lowest expectation of inflation has the highest value of the output gap and the lowest inflation of all the equilibrium points. Similarly, the equilibrium $z_*(-\rho)$ with the highest expectation of inflation has the lowest value of the output gap and the highest inflation. On the other hand, in case $c_1 < 1$, the equilibrium $z_*(\rho)$ with the highest output gap value has simultaneously the highest inflation rate.

The difference between the cases $c_1 > 1$ and $c_1 < 1$ will be further highlighted in Section 2.5.

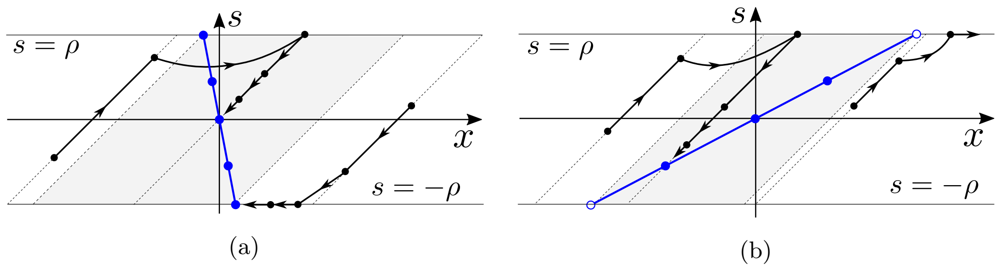


Figure 2: The projection of the line segment of equilibrium points (blue) onto the (x, s) plane for (a) $c_1 > 1$ and (b) $c_1 < 1$. The segment has a negative slope in (a) and a positive slope in (b). Sample trajectories of system (13) are shown in black.

2.4. Local stability analysis

System (7), (10) is locally linear in some neighborhood of any equilibrium point from the linear segment (15) with the exception of the two end points $z_*(\pm\rho)$ corresponding to equilibria where the play is right at one end of its inactive band. In other words, for sufficiently small deviations of the vector $z_t = (y_t, x_t)^\top$ from an interior equilibrium $z_*(s_*)$, system (13) is equivalent to

$$z_t - z_*(s_*) = B(z_t - z_*(s_*)) \tag{16}$$

where

$$B = \begin{pmatrix} \frac{1}{1+a(b_2c_1+c_2)} & \frac{a(b_1-1)c_1}{1+a(b_2c_1+c_2)} \\ \frac{b_2}{1+a(b_2c_1+c_2)} & \frac{(1-b_1)(1+ac_2)}{1+a(b_2c_1+c_2)} \end{pmatrix}$$

As shown in Appendix B, the matrix B is stable for any admissible set of parameter values, hence every equilibrium with $|s_*| < \rho$ is locally stable. This local stability ensures that if a *sufficiently small* perturbation is applied to the system residing at an equilibrium $z_*(s_*)$, removing the perturbation returns the system to the same equilibrium. Further, the eigenvalues of the matrix B determine how quickly (or slowly) the system returns to the equilibrium state. This situation is of course very similar to the expected response in a fully linear equilibrium model. The dependence of the eigenvalues of the parameters of the system is discussed in Appendix C.

However, the situation for these interior equilibria changes markedly for larger perturbations. This is related to the stability properties of the two extreme equilibria $z_*(\pm\rho)$ and is far more subtle as discussed in the next section. In particular, the basin of attraction of the equilibrium decreases and finally vanishes as one approaches either of the extreme equilibrium points along the line segment (15) (the extreme equilibria themselves are stable but not asymptotically stable).

2.5. Global stability analysis

System (13) without stickiness ($\rho = 0$) simply has the form

$$z_t = Az_{t-1}. \tag{17}$$

As shown in Appendix B, its unique zero equilibrium is globally stable if $c_1 > 1$ and is unstable if $c_1 < 1$.

For system (13) with stickiness ($\rho > 0$), equation (17) approximates the dynamics far from equilibrium points because the term s_t in (13) is bounded in absolute value by ρ. In particular, since (17) is unstable for $c_1 < 1$, so is system (13). This creates the possibility of run-away inflation at these values of c_1 (see Section 3.5).

Interestingly, the same condition $c_1 > 1$ that ensures the global stability of system (17), also guarantees the global stability of the set of equilibrium states for the sticky nonlinear system (13). In order to show this, one can use a family of *Lyapunov functions*

$$\begin{aligned} V(x_t, s_t, \nabla_t x, \nabla_t s) &= \tfrac{1}{2}\big(C(\nabla_t x)^2 + G(\nabla_t s)^2 + (Cx_t + Gs_t)^2\big) \\ &+ \gamma\big((Cx_t + Gs_t)\nabla_t x + \tfrac{H}{2C}(Cx_t + Gs_t)^2\big), \end{aligned}$$

where $\nabla_t u = u_t - u_{t-1}$, $u = x, s$. A proper choice of the parameters C, G, H, γ ensures that such a function is non-negative, achieves its minimum zero value on the linear interval of equilibrium states, and decreases to zero along every trajectory of system (13). This allows us to prove that every trajectory of system (13) converges to one of the equilibrium states (15). In the interest of space, details of the proof are omitted here and will be presented elsewhere.

For system (7) with noise, this global stability result implies that trajectories tend to return towards the segment of equilibrium points after large fluctuations and hover in a vicinity of equilibrium states for extended periods of time. The rate with which the system returns towards the line segment of equilibrium states after a large perturbation is removed is determined by the eigenvalues of the matrix A, see Appendix C.

3. Numerical results

3.1. Parameter values

The default parameter set that we use for numerical simulation is the same as in [27], see Table 1, and we shall explore in detail the surrounding parameter space. Note that,

Parameters	a	b_1	b_2	c_1	c_2
Values	0.2	0.5	0.05	1.5	0.5

Table 1: The set of parameter values.

as an example, if with the above parameters we choose $\rho = \frac{1}{2}$ then the components of the equilibrium points $z_*(s_*) = (y_*(s_*), x_*(s_*)^\top$ range over the intervals

$$y_*(s_*) \in [-5, 5], \qquad x_*(s_*) \in [-6, 6].$$

The choice of ρ is somewhat arbitrary as there is of course no corresponding reference parameter in [27] and so in many of the simulations it will be varied. Also it should be emphasized that these reference parameters are motivated by [27] but very similar numerical results were obtained for other choices.

3.2. Lower inflation volatility due to stickiness

The range of the equilibrium points of the system is directly proportional to the threshold value ρ of the play operator because the perception gap s_* in (15) can take any value in the interval $-\rho \leq s_* \leq \rho$. In particular, $\rho = 0$ corresponds to the system without stickiness in which the expectation of inflation coincides with the current inflation rate, $p = x$. This system is simply described by the equation

$$z_t = Az_{t-1} + N\xi_t \tag{18}$$

(cf. (7)). In the absence of noise, it has a unique equilibrium at $x = y = 0$.

The sticky system exhibits lower volatility in the inflation rate than the system without stickiness, see Fig. 3. This can be explained by the stability properties of matrices A and B where B is the linearization matrix of (16) for the sticky system at an equilibrium. For the parameter values of Table 1, the spectral radius of the matrix B is smaller than the spectral radius of A (see Appendix C), hence the sticky system tries to revert to equilibrium more strongly within the basin of attraction of individual equilibria, i.e. as long as the perception gap does not become extreme. Fig. 3 shows that the volatility decreases with ρ. For large (compared to ρ) deviations of z_t from the set of equilibrium points, system (7) behaves as (18).

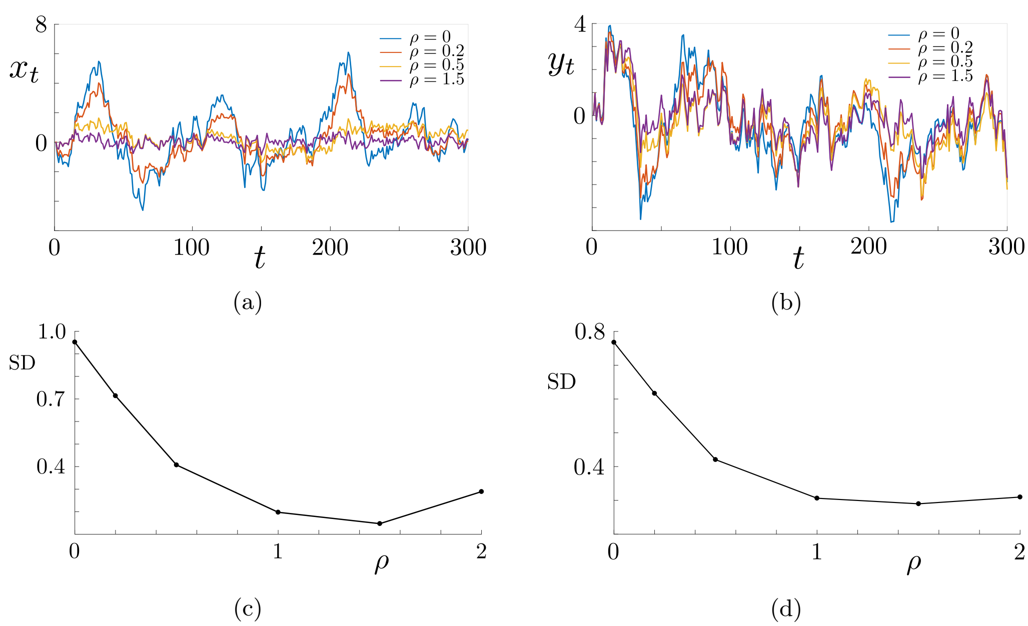


Figure 3: Trajectories of (a) inflation rate x_t and (b) output gap y_t. Measure of volatility of (c) inflation rate and (d) output gap for different values of ρ with standard deviation (SD).

3.3. Transitions between equilibrium states

The system remains within the basin of attraction of a particular equilibrium state $z_*(s_*)$ as long as the perception gap s_t does not reach either of the extreme values $\pm\rho$ and remains confined to the interval $|s_t| < \rho$, see Fig. 4(a,d). But as soon as the perception gap hits the end of its range and starts being 'dragged' by the actual inflation rate (Fig. 4(b,e)) the system transitions to the basin of attraction of a different equilibrium state where s_t becomes 'stuck' again. For this reason, the system stays near equilibrium states which correspond to non-extreme perception gaps for longer periods of time than near extreme ones. Figures 4(c,f) illustrate a transition from the equilibrium state with an extreme perception gap, $z_*(\rho)$, to one with a more moderate perception gap.

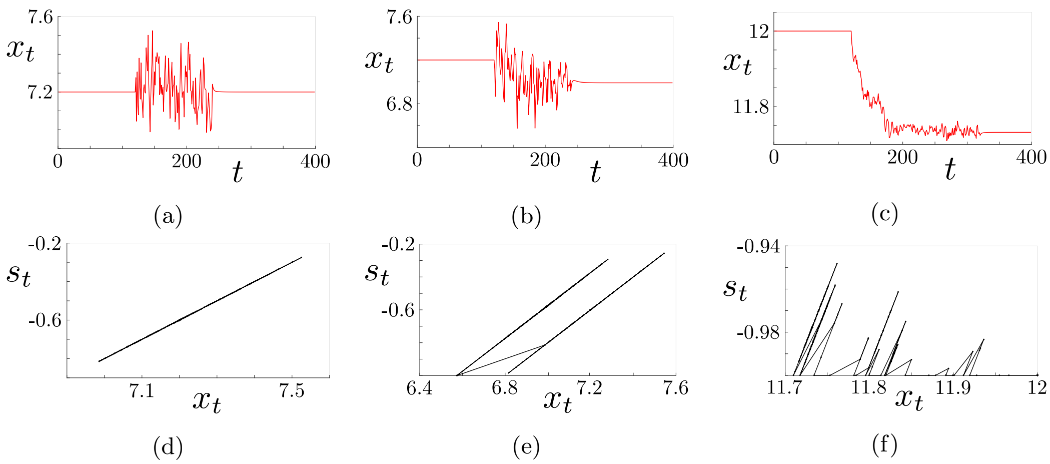
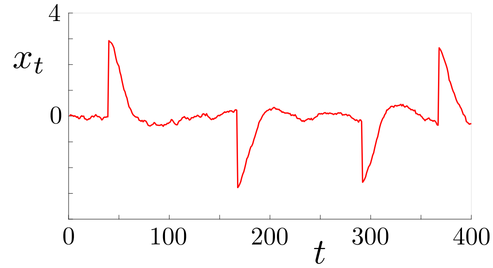


Figure 4: Transitions between the equilibrium states. (a – c) Time traces of inflation rate; (d – f) the corresponding plots in the (x, s)-space exhibiting different transition scenarios. The noise is turned off before and after the interval of time of interest in order to show the equilibrium state at the ends of this interval. (a, d) The perception gap remains within the bounds $|s_t| < \rho$, and the system stays in the basin of attraction of one equilibrium point. The inflation rate $x_*(s_*)$ is the same before and after the noisy interlude. (b, e) The perception gap reaches the extreme value $-\rho$ (the highest expectation of inflation), and the trajectory transits from the basin of attraction of an equilibrium state with higher inflation rate and lower output gap (the right slanted segment in (e)) to the basin of attraction of an equilibrium state with a lower inflation rate and higher output gap (the left slanted segment in (e)). (c, f). A transition from the equilibrium with the highest inflation rate (the rightmost point in (f)) to an equilibrium state with a more moderate inflation rate through the basins of attraction of several other equilibrium states.

3.4. Response to shocks

We shall stress the system by applying supply shocks through the term η_t. The response of the system to demand shocks applied through the term ϵ_t is similar. However, the parameter regime being considered diminishes the effect of relatively small demand shocks due to the small value of $b_2 = 0.05$.

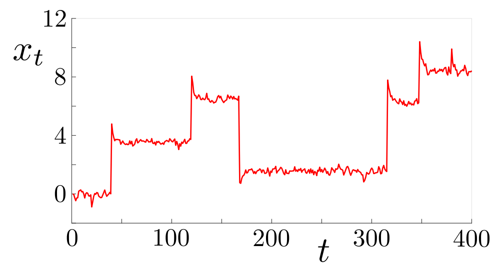
System (18) without stickiness, which has a unique globally stable equilibrium state $x_* = y_* = 0$, as expected returns to the equilibrium (and hovers near it due to noise) after each shock, see Fig. 5(a). Shocks applied to the sticky system (7), (10) result in transitions between equilibrium states, see Figure 5(b). Numerical simulation show that shocks of small magnitude typically move the system in the direction of the shock (see Fig. 6(a)). For example, after a shock that pushes up the inflation rate the system settles to a new equilibrium state, which has higher inflation rate (and lower output gap) than the equilibrium occupied prior to the shock. On the other hand, shocks of larger magnitude cause a transition to an equilibrium state that can be hard to predict because such shocks cause a longer and more complex excursion into the phase space far from equilibrium set. In Fig. 6(b), the system resides near an equilibrium with high inflation rate before a shock is



(a) (b)

Figure 5: Response to shocks. (a) The system without stickiness ($\rho = 0$) settles to the same unique equilibrium after each shock. (b) The system with stickiness ($\rho = 1$) settles to a new equilibrium after a shock is applied.

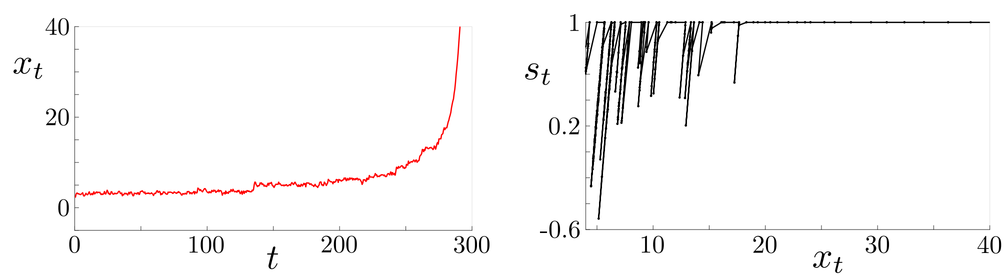
applied. Although the shock pushes the inflation even higher, the system eventually settles to an equilibrium with nearly zero inflation rate after the shock is removed.



(a) (b)

Figure 6: Response to shocks of (a) small and (b) large magnitude.

3.5. The possibility of runaway inflation



(a) (b)

Figure 7: Run-away inflation scenario. Parameter are $\rho = 1$, $a = 0.3$, $b_1 = 0.5$, $b_2 = 0.05$, $c_1 = 0.9$, $c_2 = 0.01$. The ranges of inflation rate and output gap values at equilibrium states for this set parameter are $x_* \in [-11, 11]$ and $y_* \in [-10, 10]$, respectively. (a) Time series of inflation rate x_t. (b) Trajectory in the (x, s) space.

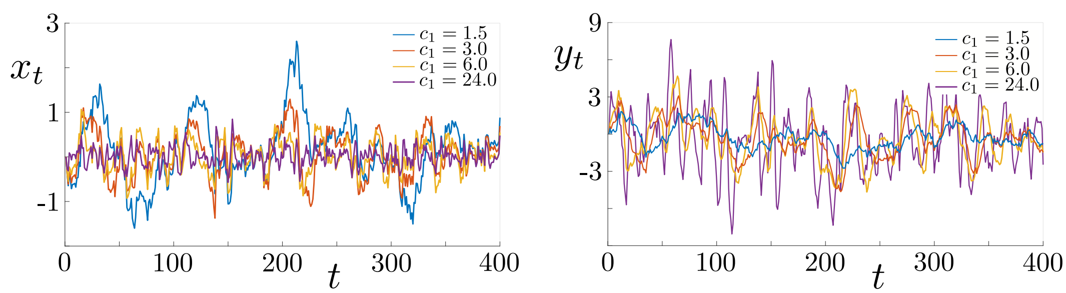
According to Section 2.5 the system is globally stable for $c_1 > 1$, but becomes unstable for $c_1 < 1$. The latter case creates a possibility of the run-away inflation scenario. It is interesting that as shown in Section 2.4 all the equilibrium points are *locally* stable even if $c_1 < 1$. As a result, dynamics appear to be stable as long as the trajectory is confined to the basin of attraction of an equilibrium state. However, when noise or a shock or another fluctuation drives the trajectory outside this bounded stability domain, the runaway scenario may and is likely to start, see Fig. 7. Just to be clear, the behavior is stable while the perception gap is not extreme, but if a shock causes that to change then the runaway instability can suddenly occur with no change in the system parameters.

3.6. A trade-off between inflation and output gap volatility

Parameters c_1 and c_2 of Taylor's rule (2) control the volatility level of inflation and output gap near an equilibrium state. Numerical simulations of the model with sticky inflation expectation show that when c_1 increases (which corresponds to stronger inflation targeting by the Central Bank), the volatility of the inflation rate decreases, see Fig. 8(a). However, at the same time, the output gap becomes highly volatile with increasing c_1, see Fig. 8(b).

When c_2 increases (stronger output gap targeting), the output gap volatility decreases, see Fig. 9(b). In particular, the case $c_2 = 0$ corresponding to pure inflation targeting in Taylor's rule is characterized by the highest volatility of the output gap. However, from Fig. 9(a), it appears that the inflation rate volatility exhibits a non-monotone behavior with c_2. This is confirmed by Fig. 10, which shows the dependence of the standard deviation of x_t and y_t on c_2 for the trajectories presented in Fig. 9. The inflation rate volatility reaches its minimum for $c_2 \approx 0.8$ for the parameter values a, b_1, b_2, c_1 from Table 1 and $\rho = 1$.

All the above results are in agreement with [27]. In addition, c_1 and c_2 affect the range of the inflation rate value at the equilibrium states for the model (7). According to (15), this range increases with c_2 and decreases with $c_1 - 1$ (for $c_1 > 1$). At the same time, the range of output gap equilibrium values is unaffected by the parameters of Taylor's rule.



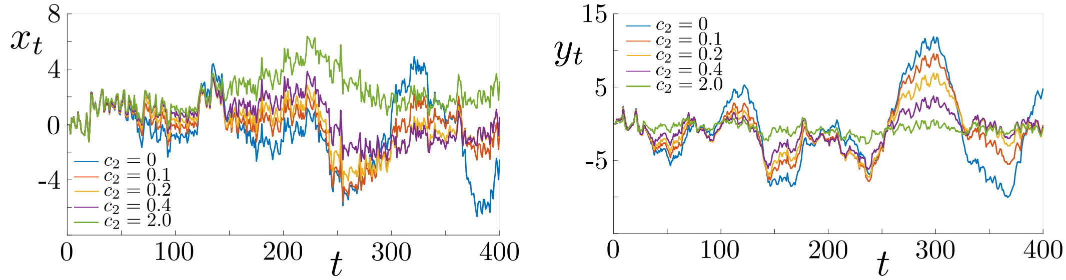
(a) (b)

Figure 8: Numerical simulations of (a) inflation rate, x_t and (b) output gap, y_t for $\rho = 1$ and various values of c_1. The remaining parameters values are from Table 1.

3.7. A multi-agent model

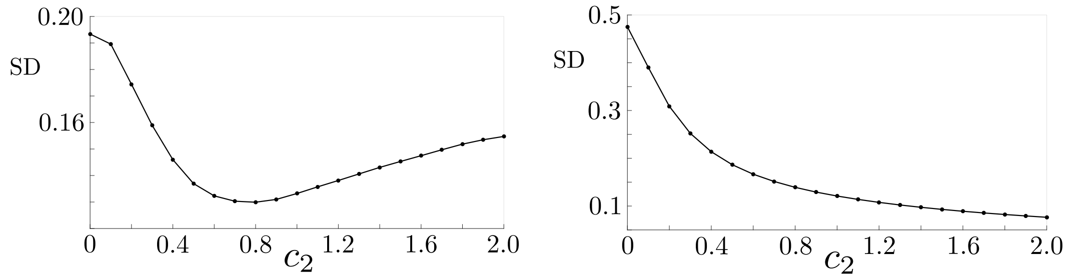
Model (7) can be easily extended to account for differing types of agent with different inflation rate expectation rules/thresholds. To this end, we replace the simple relationship (6) between p_t and x_t with the equation

$$p_t = \sum_{i=1}^{n} \mu_i \mathcal{P}_{\rho_i}[x_t] = x_t - \sum_{i=1}^{n} \mu_i \mathcal{S}_{\rho_i}[x_t] \tag{19}$$



(a) (b)

Figure 9: Numerical simulations of (a) inflation rate, x_t and (b) output gap, y_t for $\rho = 1$ and various values of c_2. The remaining parameter values are from Table 1.



(a) (b)

Figure 10: Measure of the effect of c_2 on volatility of (a) x_t and (b) y_t with standard deviation (SD).

with

$$\sum_{i=1}^{n} \mu_i = 1. \tag{20}$$

Here the play operator $\mathcal{P}_{\rho_i}$ models the expectation of inflation by the i-th agent; p_t is the aggregate expectation of inflation; $\mu_i > 0$ is a weight measuring the contribution of agent's expectation of inflation to the aggregate quantity; and, ρ_i is an individual threshold characterizing the behavior of the i-th agent. Relation (19) is equivalent to the formula

$$s_t = \mathcal{I}[x_t] := \sum_{i=1}^{n} \mu_i \mathcal{S}_{\rho_i}[x_t], \tag{21}$$

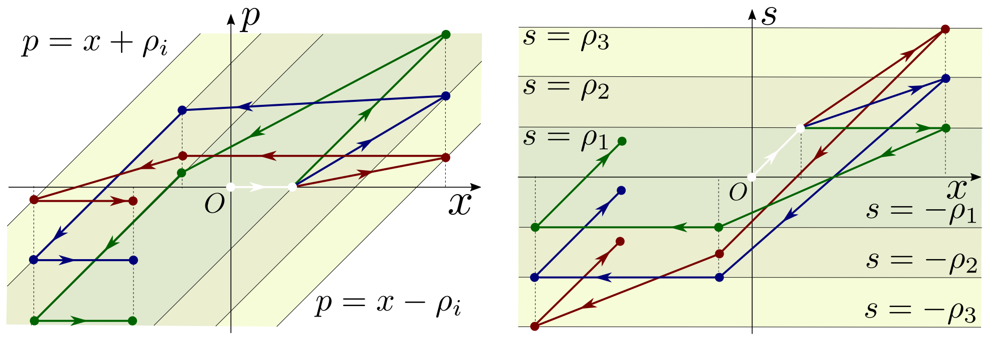
which is a (discrete) Prandtl-Ishlinskii (PI) operator with thresholds ρ_i and weights μ_i [31, 32, 33], where $s_t = x_t - p_t$.

The implicit system (1), (2), (19) with multiple agents can be converted into an explicit form using the same technique as we used for the system with one play operator. Again this involves the inversion of the PI operator. The explicit system

$$z_t = Az_{t-1} + \hat{\mathcal{I}}[c \cdot z_{t-1} + \hat{\xi}_t]\, d + N\xi_t, \tag{22}$$

which is similar to its counterpart (7), includes a PI operator with rescaled thresholds $\hat{\rho}_i$ and weights $\hat{\mu}_i$, see Appendix D for details; ξ_t, $\hat{\xi}_t$ denote the noise terms.

The stability properties of the equilibrium states of system (22) with multiple agents are similar to the stability properties considered above in Section 2.5. In particular, if we



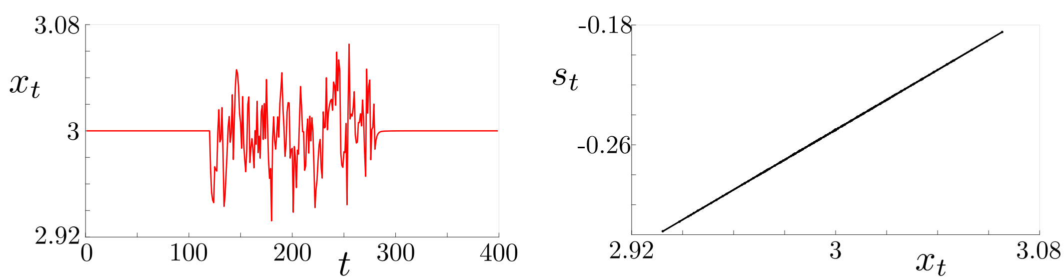
(a) (b)

Figure 11: Different expectations of agents based on three thresholds $\rho_1 < \rho_2 < \rho_3$ of (a) play and (b) stop operators with a single input x_t.

consider the system without external noise for $c_1 > 1$, then the set of equilibrium states is globally stable, and every trajectory converges to an equilibrium state.

In the simulations of this section, we classify economic agents into three categories, strongly, moderately, and weakly sensitive to inflation rate variations (hence $n = 3$), by assigning thresholds $\rho_1 < \rho_2 < \rho_3$, respectively, to these groups, see Fig. 11. Further, the contribution of each group to the aggregate expectation of inflation carries equal weight, $\mu_i = 1/3$.

Overall, numerical results obtained for model (1), (2), (19) with three agents are qualitatively similar to the results described above for the model with one agent, see Figs. 12 – 19, which are counterparts of Figs. 4 – 10, respectively.



(a) (b)

Figure 12: Trajectory of the system with 3 agents near an equilibrium state when none of the agents achieves an extreme perception gap (cf. Figure 4(a, d)). Here $c_1 > 1$. (a) Time trace of inflation. (b) Inflation versus expectation of inflation by any of the agents.

3.8. A sticky Central Bank model

The Central Bank policy can presumably exhibit stickiness too. To explore this scenario in this Section we shall replace the Taylor rule (2) with the relation

$$r_t = \mathcal{P}_\sigma[c_1 x_t + c_2 y_t] \tag{23}$$

also involving a play operator. But at the same time, for the sake of simplicity and in order to isolate the effect of stickiness in the Central Bank response upon the system, we

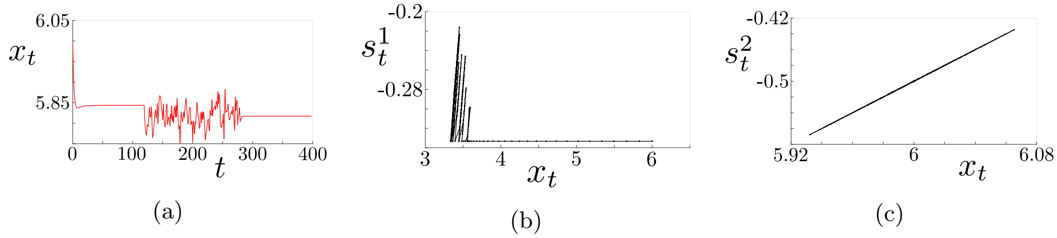


Figure 13: Trajectory of the system with 3 agents when the most sensitive agent reaches an extreme perception gap but the two less sensitive agents do not (cf. Figure 4(b, e)). The parameter c_1 satisfies $c_1 > 1$. (a) Time trace of inflation. A change of the equilibrium state occurs. (b) Inflation versus expectation of inflation by the most sensitive agent. (c) Inflation versus expectation of inflation by each of the two less sensitive agents.

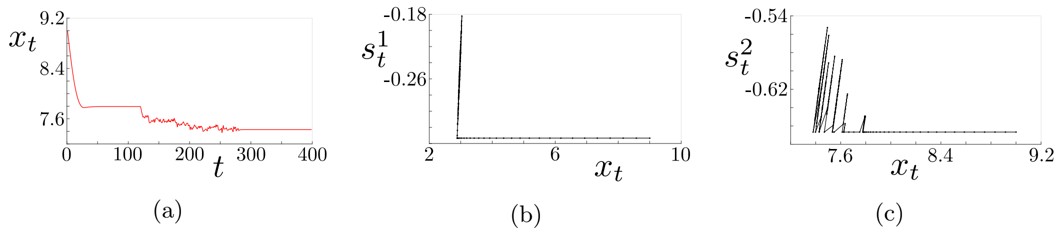


Figure 14: Trajectory of the system with 3 agents with the most sensitive agent and the moderately sensitive agent having an extreme perception gap at the initial (equilibrium) point (cf. Fig. 4(c, d)). The parameter c_1 satisfies $c_1 > 1$. (a) Time trace of inflation. (b) Inflation versus expectation of inflation for the moderately sensitive agent. (c) Inflation versus expectation of inflation for the most sensitive agent. The least sensitive agent shows the behavior as in Fig. 13(c).

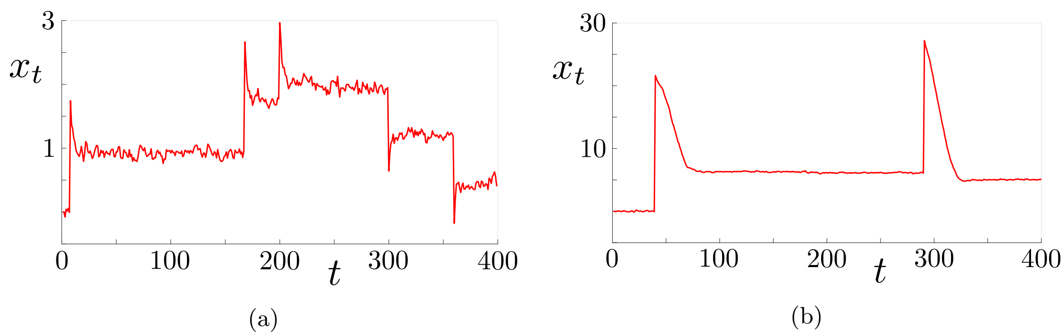


Figure 15: Changes of the equilibrium state in the model with 3 agents due to shocks (cf. Figures 5, 6). (a) Small shocks. (b) Relatively large shocks.

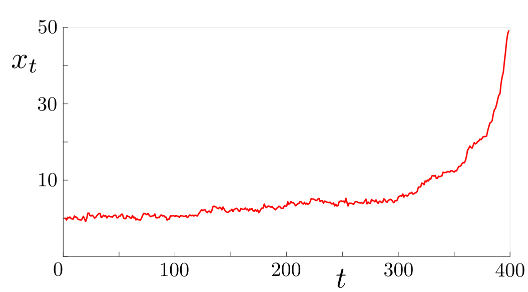
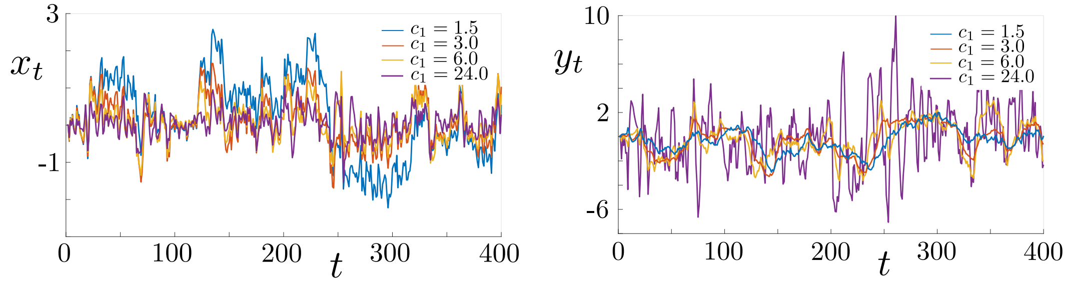
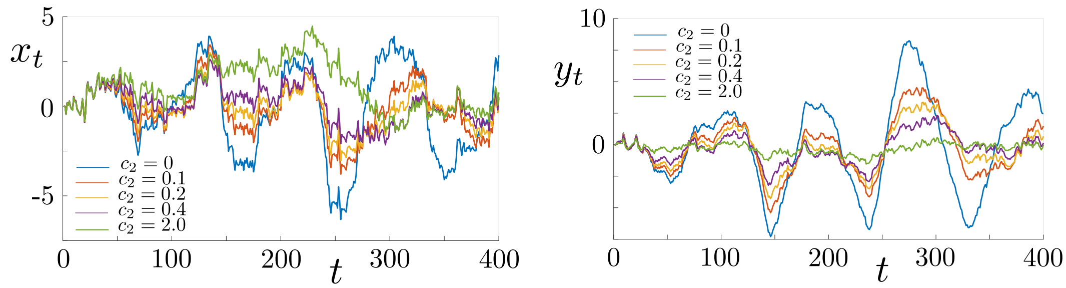


Figure 16: The run-away inflation scenario in the model with 3 agents in the case $c_1 < 1$ (cf. Fig. 7).



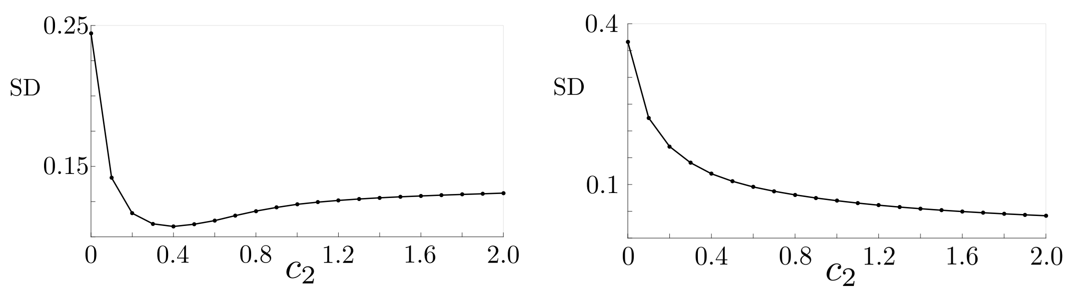
(a) (b)

Figure 17: Trade-off between the inflation and output gap volatility in the model with 3 agents as the inflation targeting parameter c_1 in the Taylor rule is varied (cf. Fig. 8). (a) Trajectories of x_t. (b) Trajectories of y_t.



(a) (b)

Figure 18: Trade-off between the inflation rate and output gap volatility in the model with 3 agents as the output gap targeting parameter c_2 in the Taylor rule is varied (cf. Fig. 9). (a) Trajectories of x_t. (b) Trajectories of y_t.



(a) (b)

Figure 19: Measure of the effect of c_2 on volatility of (a) inflation rate, x_t and (b) output gap, y_t with standard deviation (SD) (cf. Fig. 10).

remove the play operator from equations (1) thus assuming that the aggregate expectation of inflation equals to the current actual inflation rate, $p_t = x_t$; this corresponds to setting $\rho = 0$ in equations (1). In this case,

$$\begin{aligned} y_t &= y_{t-1} - a(r_t - x_t) + \epsilon_t, \\ x_t &= x_{t-1} + \tfrac{b_2}{1-b_1} y_t + \eta_t. \end{aligned} \tag{24}$$

It would be interesting to consider the model with both sticky inflation expectation and sticky Central Bank response, however this is beyond the scope of this paper.

System (23), (24) can be written in the form (7) with

$$s_t = \mathcal{S}_\sigma[c_1 x_t + c_2 y_t],$$

the matrix A defined by (8), $N = A$, and $d = (a(1-b_1), ab_2)^\top/\Delta$ with Δ defined by (9). The technique presented in Subsection 2.2 can be adapted to convert the implicit system (23), (24) into a well-defined explicit system provided that

$$1 - b_1 - ab_2 > 0. \tag{25}$$

(see Appendix E). Hence, we assume that this condition is satisfied.

Equilibrium states of system (23), (24) with zero noise terms form the line segment

$$(y_*(s_*), x_*(s_*)) = \Big(0, \frac{s_*}{c_1 - 1}\Big), \qquad s_* \in [-\sigma, \sigma]. \tag{26}$$

Notice that the output gap value is zero for all the equilibrium states, while the equilibrium inflation rate ranges over an interval of values. Notably, the local stability analysis (see Appendix E) shows that all the equilibrium states with $s_* \in (-\sigma, \sigma)$ are *unstable* for any set of parameter values. That is, stickiness in the Taylor rule leads to destabilization of equilibrium states.

On the other hand, for large values of $z_t = (y_t, x_t)^\top$, the system can be approximated by equation (17), which is exponentially stable (as shown in Appendix B). This ensures that in the system (23), (24), in the absence of noise, all trajectories converge to a bounded domain Ω surrounding the segment of equilibrium states and, upon entering this domain, remain there. However, since the equilibria are all unstable, more complicated bounded attracting orbits (such as periodic, quasiperiodic, or even chaotic attractors) must occur. Fig. 20 shows a few possibilities for the attractor of system (23), (24) obtained for different sets of parameter values. The attractor belongs to Ω whose size is controlled by the parameter σ of the sticky Taylor rule (23). This size can be estimated using the Lyapunov function introduced in Subsection 2.5.

Finally, we note that in the presence of noise, a trajectory will most likely wander unpredictable around Ω unless kicked outside temporarily by a fluctuation.

4. Conclusions

In this paper we rigorously analyzed a simple macroeconomic model using a novel form of genuinely sticky inflation expectations as opposed to the more usual 'delayed-rationality' version of stickiness. For such a simple model, defined via a single (and conceptually quite elementary) change from more standard ones, the stickiness introduces surprisingly complicated and subtle-yet-recognizable phenomena into the dynamics.

Numerically we observed: lower inflation volatility due to stickiness in inflation expectations; permanent transitions to sometimes unexpected equilibrium states due to exogenous shocks; a trade-off between inflation and output gap volatility as the targeting rule is varied, with evidence of cyclicality over long timescales; the possibility of runaway inflation due to exogenous shocks in an apparently stable system; the possibility of cascading effects in more complex models; and strong cyclicality induced by Central Bank stickiness.

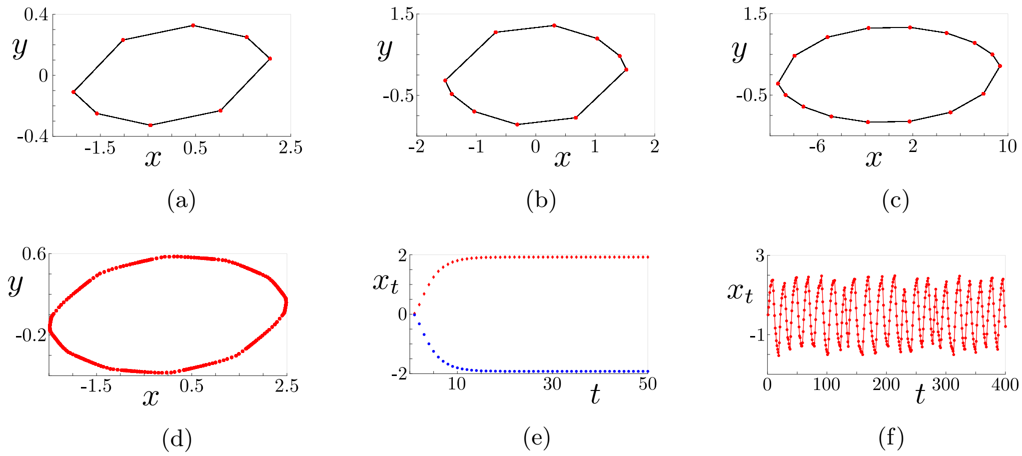


Figure 20: An attractor of system (23), (24) for several parameter sets. (a – c) A periodic orbit (with period 8, 10, 16, respectively) shown on the (x, y) plane for the system without noise. (d) A quasiperiodic orbit. (e) Two equilibrium states corresponding to $s_* = \pm\sigma$ (the time trace of x_t shown for 2 trajectories). (f) Time trace of x_t for a trajectory of the system with noise for the same parameters as in (e).

Some of the more detailed conclusions of our simulations are specific to the actual models studied but, based upon the mathematics presented here and additional numerical simulations with more complex variants of the models, we believe at least the following qualitative features to be generic and robust.

Firstly, the presence of an entire continuum of potential equilibria rather than a unique one (or even finite numbers of them). This causes permanence and path dependence at a deep level. It should be noted that in more sophisticated models, with more sticky agents and variables, the set of possible equilibria may be extremely complicated with the possibility of 'cascades' where one play operator starting to drag causes others to do so — the analogy with earthquakes made in the Introduction then becomes even closer.

Secondly, the existence of different modes depending upon whether particular sticky variables are currently stuck or being dragged. After small enough shocks the system will revert to the same equilibrium just as if it were a linear unique equilibrium model. But some modes will be less stable than others (in our main model the dragging mode is less stable than the stuck one) and a large enough shock may move the system far enough away from the set of equilibria that the route back to a new (possibly counter-intuitive) equilibrium is both long and unpredictable. An extreme example of this is when the system moves into a completely unstable regime, runaway inflation, without any change in the system parameters.

Our choice of model for a preliminary investigation into the effects of genuinely sticky economic variables was influenced by the work of De Grauwe [27] which used a different type of boundedly rational expectation formation process in a simple DSGE model. However, our genuinely sticky operators are also a viable option for modeling other sticky economic variables at both the micro- and macro-economic levels. To emphasize this, in our final model we used one to represent sticky responses by the Central Bank. The results suggest that Central Bank stickiness tends to destabilize equilibria and cause larger fluctuations in the 'Animal Spirits'.

The modeling approach presented above can be considered a 'stress test' of the usual rational expectations assumption in the underlying toy model. Or to put it another way, it is examining the robustness of an assumption rather than just the stability of the solutions within a particular model. As such, we believe that the introduction of a new kind of plausible stickiness has intrinsic merit beyond just being an alternative description of expectations formation. It provides an additional class of simple perturbations to rational models — ones that are genuinely nonlinear and capable of introducing additional phenomena in a way that merely changing the parameters of an equilibrium model cannot.

Our second and third models demonstrated that there are various ways in which this work can be extended to systems with multiple sticky agents, including the Central Bank itself, each represented by similarly sticky variables. One could also try to add aspects of adaptive learning into the agents' intrinsically sticky nature. It is of course highly unlikely that such models would be analytically tractable but if one supposes for a moment that they display similar qualitative features and adequately represent an actual economy then there are some significant policy/modeling implications.

Firstly, there is our original observation that permanence is an inherent property of the system. After sufficiently small shocks the system returns to the same equilibrium but after larger shocks it will not. This does not mean however that the model parameters have changed. Indeed changing the parameters in a unique equilibrium model to match and then try to control a path-dependent reality may well introduce additional instabilities. This would be an interesting line of research.

Secondly, different path-dependent equilibria have different stability properties and those close to the boundary of the set of feasible equilibria are typically only marginally stable. So the system's equilibrium may move around the set of feasible equilibria for a very long time until suddenly everything changes. Either a seemingly unremarkable exogenous shock or a sudden cascade of endogenous sticky variables changing their mode take the system on a far-from-equilibrium (but still bounded!) excursion with a very unpredictable outcome somewhere back on the set of feasible equilibria. None of this need involve any change in the system parameters and so even long periods of stability should not lead to complacency.

Note that both of these final observations are made in generality (not specifically referring to sticky inflation expectations) and may be compared with the quote from the Introduction, written shortly after the 'Great Moderation' finally ended in 2007/8. We hope that this paper will help stimulate further theoretical results concerning qualitative changes resulting from the introduction of bounded rationality into rational, unique equilibrium, models.

Appendix

A. Derivation of equations (7), (10)

Here we show how to obtain equations (7), (10) from model (1)–(4). To this end, we substitute the equation for r_t into the equation for y_t and obtain

$$(1 + ac_2)y_t = y_{t-1} - ac_1x_t + ap_t + \epsilon_t.$$

Next, we substitute this equation into the equation for x_t and simplify to obtain

$$\gamma x_t - \beta p_t = b_2 y_{t-1} + (1-b_1)(1+ac_2)x_{t-1} + b_2\epsilon_t + (1+ac_2)\eta_t, \tag{27}$$

where

$$\gamma = 1 + ac_2 + ab_2c_1, \qquad \beta = b_1(1+ac_2) + ab_2.$$

Since $p_t = x_t - s_t$, equation (27) can be rewritten as

$$\alpha x_t + s_t = f_t \tag{28}$$

with α and f_t defined by (11), (14). Therefore, $x_t = \alpha^{-1}(f_t - s_t)$, which combined with (11), (14) gives

$$x_t = \frac{b_2}{\alpha\beta}y_{t-1} + \frac{(1-b_1)(1+ac_2)}{\alpha\beta}x_{t-1} - \frac{1}{\alpha}s_t + \frac{b_2}{\alpha\beta}\epsilon_t + \frac{1+ac_2}{\alpha\beta}\eta_t. \tag{29}$$

Subsequently, substituting equation (29) into equation (4) gives

$$\begin{aligned} y_t \;\;=\;\; & \frac{ab_2(1-c_1)+\alpha\beta}{\alpha\beta(1+ac_2)}y_{t-1} + \frac{a(1-c_1)(1-b_1)}{\alpha\beta}x_{t-1} \\ & + \frac{a(c_1-1-\alpha)}{\alpha(1+ac_2)}s_t + \frac{\alpha\beta + ab_2(1-c_1)}{\alpha\beta(1+ac_2)}\epsilon_t + \frac{a(1-c_1)}{\alpha\beta}\eta_t. \end{aligned} \tag{30}$$

Equations (29), (30) can be written as system (7) with the matrices A, N and the vector d defined by formulas (8).

Equation (10) can be obtained from relation (28) using the inversion formula for the play operator. This inversion formula is presented for a more general Prandtl-Ishlinskii (PI) operator, including the play operator as a particular case, in Appendix D.

B. Local stability analysis

The characteristic polynomial of matrix B is

$$P_B(\lambda) = \lambda^2 - \lambda\left(\frac{2+ac_2-b_1(1+ac_2)}{1+a(b_2c_1+c_2)}\right) + \frac{1-b_1}{1+a(b_2c_1+c_2)}.$$

Applying Jury's stability criterion to the characteristic polynomial gives the following set of inequalities:

$$\begin{aligned} P_B(1) \;&=\; 1 - \frac{2+ac_2-b_1(1+ac_2)}{1+a(b_2c_1+c_2)} + \frac{1-b_1}{1+a(b_2c_1+c_2)} > 0, \\ P_B(-1) \;&=\; 1 + \frac{2+ac_2-b_1(1+ac_2)}{1+a(b_2c_1+c_2)} + \frac{1-b_1}{1+a(b_2c_1+c_2)} > 0, \\ 1 \;&>\; \frac{1-b_1}{1+a(b_2c_1+c_2)}. \end{aligned}$$

It is easy to see that all the three inequalities above are satisfied for any set of parameters $a, b_2, c_1, c_2 > 0$ and $0 < b_1 < 1$, hence every equilibrium $z_*(s_*)$ with $|s_*| < \rho$ is locally stable.

Now, let us consider the system without stiction. The characteristic polynomial of matrix A is

$$P_A(\lambda) = \Delta\lambda^2 - (1-b_1)(2+ac_2)\lambda + 1 - b_1$$

with Δ defined by (9). Applying Jury's stability criterion, we obtain

$$\begin{aligned} P_A(1) &= 1 - \frac{(1-b_1)(2+ac_2)}{\Delta} + \frac{1-b_1}{\Delta} > 0, \\ P_A(-1) &= 1 + \frac{(1-b_1)(2+ac_2)}{\Delta} + \frac{1-b_1}{\Delta} > 0, \\ 1 &> \frac{1-b_1}{\Delta}. \end{aligned}$$

Taking into account the constraints $a, b_2, c_1, c_2 > 0$ and $0 < b_1 < 1$, these conditions result in the relationship

$$c_1 > 1.$$

Note that the system $z_t = Az_{t-1}$ is the linearization of sticky system (7) at infinity, hence it describes the return of the sticky system towards near equilibrium dynamics after a large perturbation. Thus, the stability condition $c_1 > 1$ for A agrees with the global stability criterion obtained in Section 2.5.

C. *The effect of parameters on stability properties*

Here we provide some numerical analysis concerning the effect of the parameters on stability properties of the equilibrium states. Stronger stability generally implies lower volatility and more infrequent transitions between different equilibrium states. We quantify local stability using the maximum absolute value, $|\lambda_{i,e}|$, of eigenvalues of the linearized system at an equilibrium point. The subscripts e and i refer to the system without stickiness ($\rho = 0$) and with stickiness ($\rho = 1$), respectively.

The model contains five other parameters, a, b_1, b_2, c_1 and c_2. Fig. 21 shows the dependence of $|\lambda_{i,e}|$ on the parameter a and implies that the system with stickiness is more stable than the system without stickiness. Other parameter values are taken from Table 1. Interestingly, the system with stickiness becomes more stable for increasing a, while this dependence for the non-sticky system is non-monotone since $|\lambda_e|$ has a minimum at $a \approx 0.8$.

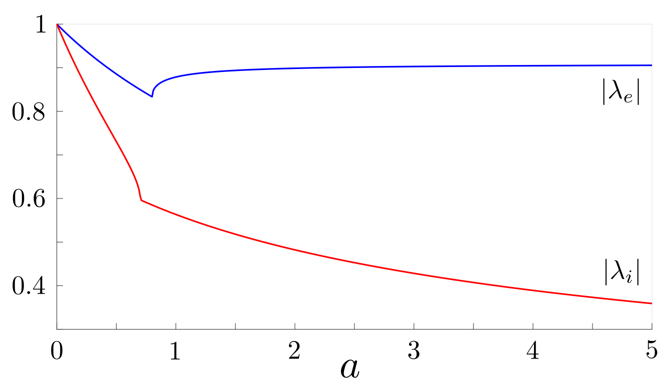
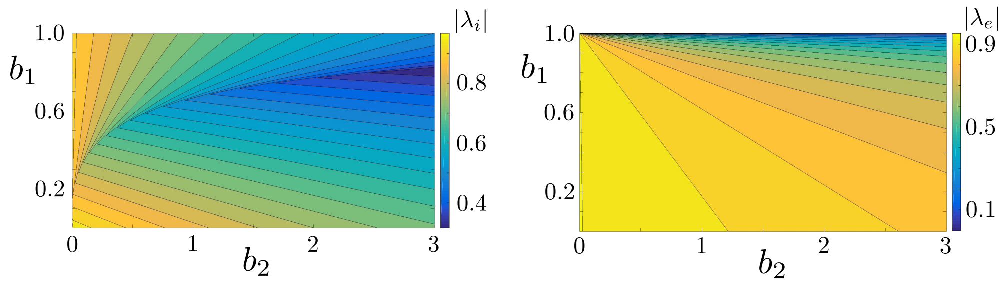


Figure 21: Variation of $|\lambda_i|$ and $|\lambda_e|$ with a. Other parameters are taken from Table 1.

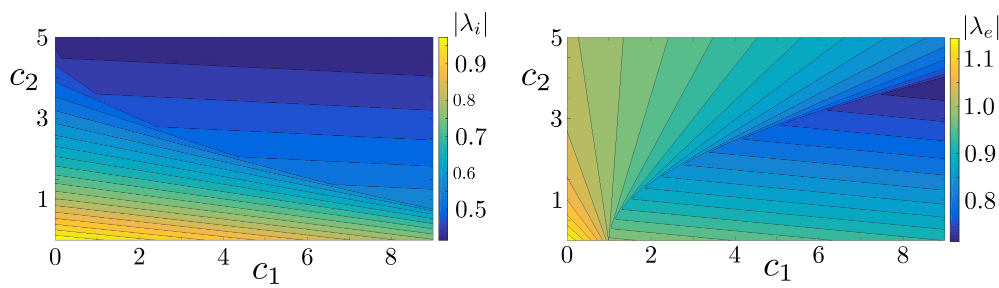
The range of output gap equilibrium values is proportional to the ratio of parameters b_1 and b_2 according to (15). Fig. 22 presents the dependence of $|\lambda_{i,e}|$ on these parameters. The sticky system is more stable than its non-sticky counterpart for $b_1 < 0.9$, but becomes less stable than the non-sticky system as b_1 approaches 1 (in the latter case, the future inflation rate is defined predominantly by expectations). The dependence of $|\lambda_{i,e}|$ on b_2 and the dependence of $|\lambda_e|$ on b_1 is monotone (stronger stability for larger $b_{1,2}$), while the dependence of $|\lambda_i|$ on b_2 is non-monotone. The strongest stability is achieved by the sticky system for some intermediate value of b_1 between 0 and 1.



(a) (b)

Figure 22: Dependence of (a) $|\lambda_i|$ and (b) $|\lambda_e|$ on b_1 and b_2. Other parameters are taken from Table 1.

Parameters c_1 and c_2 control the range of inflation rate equilibrium values according to (15). This range contracts when c_1 increases (for $c_1 > 1$) and expands when c_2 increases. Fig. 23 shows that the sticky system is generally more stable than the non-sticky one. Both systems become more stable with increasing c_1 (stronger inflation targeting in Taylor's rule), see Figs. 23(a, b) and 24(a, b). The dependence of $|\lambda_i|$ on c_2 demonstrates some slight non-monotonicity for large c_2 values, see Figure 24(b). The non-monotonicity of $|\lambda_i|$ with c_2 is much more pronounced with the minimum achieved for a certain value of c_2 depending on c_1, see Figs. 23(b) and 24(b). This minimum corresponds to the strongest stability and, in this sense, optimizes the Central Bank policy. In Fig. 23(b), the strongest stability is achieved on the 'parabolic' line.



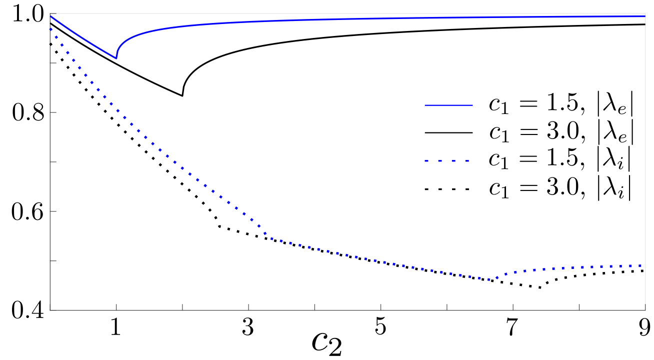
(a) (b)

Figure 23: Dependence of (a) $|\lambda_i|$ and (b) $|\lambda_e|$ on c_1 and c_2. Other parameters are taken from Table 1.

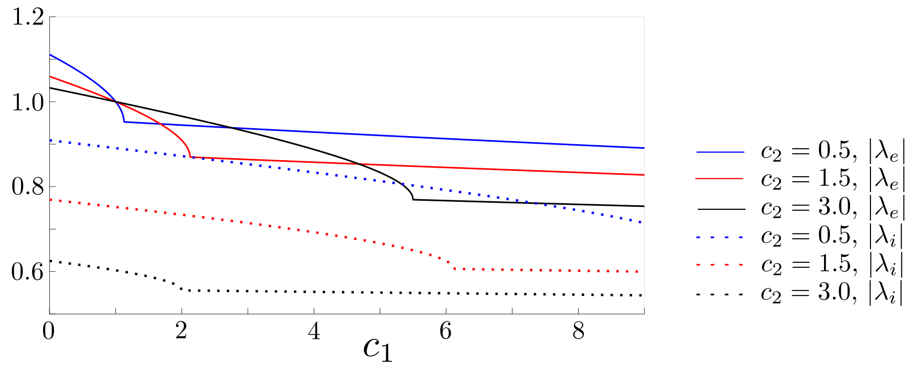
D. Inversion of the PI operator

In this section, we consider the inversion of the PI operator, which is necessary to transform the implicit system (1), (2) coupled with relation (19) into the explicit form (22). Here we use the term 'PI operator' for an input-output relationship of the form

$$f_t = \alpha x_t + \sum_{i=1}^{n} \mu_i \mathcal{S}_{\rho_i}[x_t], \tag{31}$$



(a)



(b)

Figure 24: Cross-sections of the plots shown in Fig. 23 (a) for various c_2 values and (b) for various c_1 values.

where the weights μ_i are allowed to have any sign, $\alpha \geq 0$, and $\rho_1 < \rho_2 < \cdots < \rho_n$. Such an operator is completely defined by the so-called *Primary Response* (PR) *function* $\phi(x)$, which describes the output in response to a monotonically increasing input. Here, this is a

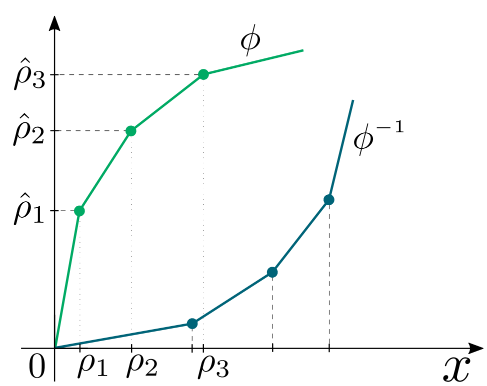


Figure 25: PR function ϕ of PI operator (31) and PR function ϕ^{-1} of its inverse PI operator (32).

piecewise linear continuous function satisfying $\phi(0) = 0$ with the slopes defined by

$$\phi'(x) = \begin{cases} \alpha + \mu_n + \cdots + \mu_2 + \mu_1, & 0 < x < \rho_1, \\ \alpha + \mu_n + \cdots + \mu_2, & \rho_1 < x < \rho_2, \\ \vdots & \\ \alpha + \mu_n, & \rho_{n-1} < x < \rho_n, \\ \alpha, & x > \rho_n, \end{cases}$$

see Fig. 25. As shown in [34], if the slopes of ϕ are all positive, then the PI operator (31) is invertible, and the inverse relationship is also a PI operator:

$$x_t = \hat{\alpha} f_t + \sum_{i=1}^{n} \hat{\mu}_i \mathcal{S}_{\hat{\rho}_i}[f_t]. \tag{32}$$

Further, the PR function of operator (32) is the inverse of the PR function ϕ of operator (31). This allows one to express the weights $\hat{\alpha}, \hat{\mu}_i$ and the thresholds $\hat{\rho}_i$ explicitly in terms of the weights α, μ_i and the thresholds ρ_i. In particular, the equation $\alpha x_t + s_t = f_t$ with $s_t = \mathcal{S}_\rho[x_t]$ (see (28)) can be inverted as

$$x_t = \frac{1}{\alpha} f_t - \frac{1}{\alpha(1+\alpha)} \mathcal{S}_{(1+\alpha)\rho}[f_t],$$

and this implies $s_t = \frac{1}{1+\alpha} \mathcal{S}_{(1+\alpha)\rho}[f_t]$, which is equivalent to (10) (cf. Appendix A).

E. Sticky Taylor rule

In order to convert system (23), (24) to the explicit form, we replace the variable y_t with the variable $g_t = c_1 x_t + c_2 y_t$ and obtain

$$g_t = (c_1 + a c_2) x_t + g_{t-1} - c_1 x_{t-1} - a c_2 \mathcal{P}_\sigma[g_t] + c_2 \epsilon_t, \tag{33}$$

$$x_t = \frac{c_2(1-b_1)}{b_2 c_1 + c_2(1-b_1)} x_{t-1} + \frac{b_2}{b_2 c_1 + c_2(1-b_1)} g_t + \frac{c_2(1-b_1)}{b_2 c_1 + c_2(1-b_1)} \eta_t. \tag{34}$$

Further, substituting (34) into (33) gives

$$\alpha g_t + \kappa \mathcal{P}_\sigma[g_t] = f_t \tag{35}$$

with

$$\alpha = \frac{c_2(1 - b_1 - a b_2)}{b_2 c_1 + c_2(1-b_1)}, \qquad \kappa = a c_2,$$

$$f_t = g_{t-1} - c_1 x_{t-1} + \frac{c_2(1-b_1)(c_1 + a c_2)}{b_2 c_1 + c_2(1-b_1)} (x_{t-1} + \eta_t) + c_2 \epsilon_t.$$

Using that $\alpha > 0$ due to (25), we can invert (35) as in Appendix D to obtain

$$g_t = \frac{1}{\alpha}\left(f_t - \frac{\kappa}{\alpha+\kappa} \mathcal{P}_{\alpha\sigma}[f_t] \right).$$

This equation together with (34) defines the explicit system for (23), (24). The linearization $z_t = B z_{t-1}$ of this system at any equilibrium point with $s_* \in (-\sigma, \sigma)$ has the matrix

$$B = \frac{1}{1 - b_1 - a b_2} \begin{pmatrix} 1 - b_1 & a(1-b_1) \\ b_2 & 1 - b_1 \end{pmatrix}.$$

Since

$$\det B = \frac{1 - b_1}{1 - b_1 - a b_2} > 1,$$

all these equilibrium states are unstable.

Acknowledgments

DR and EK acknowledge the support of NSF through grant DMS-1413223. PK acknowledges the support of GAČR through grant GA15-12227S.

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