# On the periodically perturbed logistic equation

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**Abstract**

We study the logistic equation modified by a periodic time-dependence. The perturbation introduces bifurcation delays which can be calculated explicitly and we qualitatively explain how the bifurcation diagram deforms as the perturbation is increased.

# 1 Introduction

The logistic equation $x_{n+1} = \lambda x_n(1-x_n)$ has been used for almost twenty years as a tool for investigating deterministic chaos in dynamical systems; [6] has a good account.

Somewhat more tenuously, the logistic equation is presumed to model single species population dynamics under the assumption that generations do not overlap [8]. Once the ecological rationale for using the logistic equation is accepted, various modifications suggest themselves, for example to systems consisting of more than one species with various modes of interaction [6, 10]. Another approach is to suggest that the species reproduces more than once a year and that its intrinsic fertility rate $\lambda$ is season–dependent, i.e. that $\lambda \equiv \lambda_n$, a periodic function of $n$. The simplest such case would be when it has period two, so that we have

$$x_{n+1} = (\lambda + (-1)^n \epsilon) x_n (1 - x_n). \tag{1.1}$$

This is the equation we study in this note. Without loss of generality we take $\epsilon \geq 0$. Time–dependent fertility rates have been considered previously, for example, by [7, 1, 3]. In [7] the parameter $\lambda$ is a monotone increasing function of $n$ (which has no obvious biological justification); in [1] the parameter is subjected to both sweep and noise. In [3] addition of white noise to the right–hand side of the logistic equation is considered; this also does not admit a biological interpretation. We also mention [11], where influence of seasonal periodicity on predator–prey equations is studied mainly by numerical means.

Surprisingly, the simple equation (1.1) still has much to offer. In particular, it exhibits bifurcation delays which can be quantified exactly, coexistence of stable attractors and attractor crises. In our opinion, the influence of $\epsilon \neq 0$ on the bifurcation scenario is not at all obvious, and that is justification enough for studying it.

# 2 Structure of the initial bifurcations

First of all note that apart from $x = 0$, (1.1) has no stationary points. The solution $x = 0$ loses stability when $\lambda^2 - \epsilon^2 = 1$. If $\lambda + \epsilon > 4$, almost all points map sooner or later out of the interval. so that in order to have any nontrivial dynamics, we must assume that $4 - \epsilon > \sqrt{1 + \epsilon^2}$, that is, that $\epsilon < 15/8$.

It makes sense to assume initially that $\epsilon$ is very small. In short, the influence of $\epsilon \neq 0$ small on the first two bifurcations in the period doubling cascade is very different. Instead of the first pitchfork we have two fold (tangent) bifurcations. The result is that, unlike the bifurcation from a stable fixed point to a stable two–cycle as in the logistic equation, we have coexistence of two stable two–cycles after the fold bifurcation (coexistence of stable attractors is impossible for the logistic equation). The second pitchfork bifurcation splits into two, one for each two–cycle. Thus, instead of the usual Sharkovskii sequence $1 \mapsto 2 \mapsto 4 \mapsto \ldots$, we have $2 \mapsto 2 \times 2 \mapsto 2 \times 4 \mapsto \ldots$, where $\times$ stands for coexistence of stable cycles.

We will show analytically that this indeed is the case. The demonstration also indicates how one would expect the bifurcation diagrams to deform as $\epsilon$ grows.

To understand the bifurcations, we need the equivalent of the twice–iterated map of the logistic equation. Obviously, there are two such maps in this case. Thus, if

$$f_1(x) \equiv (\lambda + \epsilon)x(1 - x) \text{ and } f_2(x) \equiv (\lambda - \epsilon)x(1 - x),$$

we can define $f_{12}(x) \equiv f_2(f_1(x))$ and $f_{21}(x) \equiv f_1(f_2(x))$. The chain rule tells us that their bifurcations occur at the same values of parameters. More precisely, if $x_0$ is a fixed point of $f_{21}$, then $f_2(x_0)$ is a fixed point of $f_{12}(x)$. Furthermore, $f'_{21}(x_0) = f_{12}(f'_2(x_0))$. To see this, note that

$$f'_{12}(x)|_{x=f_2(x_0)} = f'_2(f_1(x)) \cdot f'_1(x)|_{x=f_2(x_0)} = f'_2(x) \cdot f'_1(f_2(x))|_{x=x_0} = f'_{21}(x)|_{x=x_0}.$$

Therefore it suffices to consider the bifurcations of one of them, say $f_{12}(x)$. We are particularly interested in the $\epsilon$–dependence of the values of $\lambda$ for which various bifurcations occur.

Before that, however, we can apply some powerful theorems to prove results concerning coexistence of attracting solutions and basins of attraction for (1.1). Let us consider the map $f_{21}$. Then $f'_{21}(x) = f'_1(f_2(x)) \cdot f'_2(x)$ and so if $\lambda - \epsilon < 2$ the only critical point is $x = 1/2$, otherwise there are two more critical points given by $x = f_2^{-1}(1/2)$. It is easy to check that each of these critical points is non-flat, that is at least one derivative is non-zero at each of these points. In [4] it is proved that if $f : I \to I$ is a $C^2$ map with non-flat critical points then $f$ has no wandering intervals. Thus $f_{21}$ (and hence $f_{12}$) has no wandering intervals. In [4] the authors is also prove a generalisation of Singer's Theorem: if $f$ as above is $C^3$, has no wandering intervals, and has negative Schwartzian derivative then each attractor of $f$ has a critical point or a boundary point in its basin of attraction. The function $f_{21}$ has negative Schwartzian derivative since both $f_1$ and $f_2$ have this property which is conserved under composition and so the conditions hold.

The boundary points 0 and 1 both map into 0 and for $\lambda - \epsilon > 2$ both the critical points given by $x = f_2^{-1}(1/2)$ map to $f_1(1/2)$. Therefore we have the result that $f_{21}$ (and therefore $f_{12}$) has precisely one attractor for $\lambda - \epsilon < 2$ and at most two attractors otherwise (apart from possibly the point 0). Also, all the attractors can be found by iterating the point $x = 1/2$ from $n$ even and $n$ odd.

## 2.1 The tangent bifurcation

At the first tangent bifurcation we must have simultaneously

$$f_{12}(x) = x \text{ and } f'_{12}(x) = 1.$$

This is a system of two polynomial equations in $x$, $\lambda$ and $\epsilon$. As we are not interested in the value of $x$ for which the bifurcation occurs, we eliminate $x$ from the equations using resultants (Gröbner bases could be used as well). The computation (it and all other computations were performed using MAPLE V [2]) of the resultant of two polynomials with respect to $x$, leads after factorization and picking the correct factor, to

$$r_1(\lambda,\ \epsilon) = \epsilon^4 + 8\epsilon^2\lambda - 18\epsilon^2 - 2\lambda^2\epsilon^2 + 18\lambda^2 - 8\lambda^3 + \lambda^4 - 27 = 0.$$

This is the relation between the value of $\lambda$, $\lambda_t(\epsilon)$ and $\epsilon$ at the tangent bifurcation. It has to be solved numerically for values of $\epsilon$ in an interval to be discussed later. The expression for $r_1(\lambda,\ \epsilon)$ can be further simplified by noting that $\lambda_t(0) = 3$ (when $\epsilon = 0$ the two tangent bifurcations collide to give the first pitchfork of the logistic equation), so on setting $\lambda = \mu + 3$, we have

$$\epsilon^4 - 4\epsilon^2\mu - 12\epsilon^2 - 2\epsilon^2\mu^2 + 4\mu^3 + \mu^4 = 0.$$

For small values of $\epsilon$ we can use the Newton polygon procedure to come up with the asymptotic expansion

$$\lambda_t(\epsilon) = 3 + 3^{1/3}\epsilon^{2/3} + \mathcal{O}(\epsilon^{4/3}).$$

Note that we have here bifurcation delay due to the perturbation. It is natural to try to understand for what value of $\epsilon$ the maximal bifurcation delay is achieved.

## 2.2 The pitchfork bifurcations

To answer that question, we must consider what happens to the second pitchfork bifurcation of the logistic equation under a period two perturbation. Again, the answer is furnished by $f_{12}(x)$. At a pitchfork bifurcation we have

$$f_{12}(x) = x \text{ and } f'_{12}(x) = -1.$$

Proceeding as before, the resultant of the two equations with respect to $x$ is

$$r_2(\lambda,\ \epsilon) = -125 - 85\epsilon^2 - 15\epsilon^4 + \epsilon^6 + 24\epsilon^2\lambda + 85\lambda^2 + 30\lambda^2\epsilon^2 - \quad (2.2)$$
$$3\lambda^2\epsilon^4 - 24\lambda^3 - 16\lambda^3\epsilon^2 - 15\lambda^4 + 3\lambda^4\epsilon^2 + 8\lambda^5 - \lambda^6 = 0. \quad (2.3)$$

This then is the relation between $\lambda_p(\epsilon)$, the value of $\lambda$ at which a two–cycle loses stability to a four–cycle. Note that at least for small enough $\epsilon$ there are two such bifurcations: one from the original period two solution (which just reflects the fact that the equation (1.1) is periodic in $n$ with period two) and one from the two–cycle generated by the tangent bifurcation discussed above. Let us denote the values of $\lambda$ for which these bifurcations occur by $\lambda_p^1(\epsilon)$ and $\lambda_p^2(\epsilon)$. Obviously, $\lambda_p^1(0) = \lambda_p^2(0) = 1 + \sqrt{6}$. $r_2(\lambda,\ \epsilon)$ is too cumbersome to be further processed analytically and is solved numerically. We present the graphs of $\lambda_p^1(\epsilon)$ , $\lambda_p^2(\epsilon)$ and $\lambda_t(\epsilon)$ in Figure 1. (All the pictures in this note have been generated using MATLAB [9]).

**Insert Figure 1 here**

Note that $\lambda_t(\epsilon)$ is a monotone increasing function of $\epsilon$, as can be seen from implicitly differentiating $r_1(\lambda(\epsilon),\ \epsilon) = 0$ in $\epsilon$. From the numerics, $\lambda_p^2(\epsilon)$ is also monotone increasing, while $\lambda_p^1(\epsilon)$ is monotone decreasing. In particular, for $\epsilon > 0.34861191$, which solves $r_2(3,\ \epsilon) = 0$ we have that $\lambda_p^1 < 3$, so instead of having bifurcation delay, we have, so to speak, "premature bifurcation".

Now it becomes clear that bifurcation delay is maximal when $\lambda_p^1(\epsilon) = \lambda_t(\epsilon)$. To obtain that value of $\epsilon$, we take the resultant of $r_1(\lambda,\ \epsilon)$ and $r_2(\lambda,\ \epsilon)$ with respect to $\lambda$. The result is very simple:

$$20480 - 1982464\epsilon^2 = 0,$$

which we solve for $\epsilon$ to obtain the maximal delay value, $\epsilon_d = 0.1016394$.

In addition, we will have coexistence of two attractors as long as $\lambda_t(\epsilon) < 4 - \epsilon$. To obtain that value we simply solve $r_1(4-\epsilon,\ \epsilon) = 0$. This collapses to $5 - 16\epsilon = 0$, which gives us the maximal value for coexistence, $\epsilon_c = 0.3125$.

# 3 Bifurcation diagrams

Thus it would appear from the above analysis of the (at most three) initial bifurcations that the bifurcation diagrams of (1.1) fall into three classes: (a) $0 < \epsilon < \epsilon_d$; (a) $\epsilon_d < \epsilon < \epsilon_c$; (c) $\epsilon > \epsilon_c$. Typical examples are shown in Figure 2. We make the obvious point that due to the non–autonomous nature of the equation, branches can intersect, both if they belong to the same attractor (in which case the trajectory waits for a unit of time at the intersection point) or if they belong to different attractors; this means that trajectory of the intersection point depends on whether $f_1$ or $f_2$ is to be applied to it.

**Insert Figure 2 here**

Finally, we also draw the attention of the reader to attractor crises and reappearance. This obviously has to do with collision of a chaotic attractor with an unstable invariant set [5]. An example is shown in Figure 2(a).

# References


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# Captions for Figures

Figure 1: The functions $\lambda_p^1(\epsilon)$ (solid line), $\lambda_p^2(\epsilon)$ (dashed line), and $\lambda_t(\epsilon)$ (dotted line).

Figure 2: Three types of bifurcation diagrams. (a) $\epsilon = 0.08$; (b) $\epsilon = 0.2$, (c) $\epsilon = 0.38$.