The Transition from Brownian Motion to Boom-and-Bust Dynamics in Financial and Economic Systems

Harbir Lamba

Abstract Quasi-equilibrium models for aggregate or emergent variables over long periods of time are widely-used throughout finance and economics. The validity of such models depends crucially upon assuming that the system participants act both independently and without memory. However important real-world effects such as herding, imitation, perverse incentives and many of the key findings of behavioral economics violate one or both of these key assumptions.

We present a very simple, yet realistic, agent-based modeling framework that is capable of simultaneously incorporating many of these effects. In this paper we use such a model in the context of a financial market to demonstrate that herding can cause a transition to multi-year boom-and-bust dynamics at levels far below a plausible estimate of the herding strength in actual financial markets. In other words, the stability of the standard (Brownian motion) equilibrium solution badly fails a 'stress test' in the presence of a realistic weakening of the underlying modeling assumptions.

The model contains a small number of fundamental parameters that can be easily estimated and require no fine-tuning. It also gives rise to a novel stochastic particle system with switching and re-injection that is of independent mathematical interest and may also be applicable to other areas of social dynamics.

1 Introduction

In the physical sciences using a stochastic differential equation (SDE) to model the effect of exogenous noise upon an underlying ODE system is often straightforward. The noise consists of many uncorrelated effects whose cumulative impact is well-

Harbir Lamba
Department of Mathematical Sciences, George Mason University, Fairfax, USA., e-mail: hlamba@gmu.edu

approximated by a Brownian process $B_s,\ s \geq 0$ and the ODE $df = a(f,t)\ dt$ is replaced by an SDE $df = a(f,t)\ dt + b(f,t)\ dB_t$.

However, in financial and socio-economic systems the inclusion of exogenous noise (ie new information entering the system) is more problematic — even if the noise itself can be legitimately modeled as a Brownian process. This is because such systems are themselves the aggregation of many individuals or trading entities (referred to as *agents*) who typically

a) interpret and act differently to new information,
b) may act differently depending upon the recent system history (ie non-Markovian behaviour), and
c) may not act independently of each other.

The standard approach in neoclassical economics and modern finance is simply to 'average away' these awkward effects by assuming the existence of a single representative agent as in macroeconomics [7], or by assuming that the averaged reaction to new information is correct/rational, as in microeconomics and finance [13, 4]. In both cases, the possibility of significant endogenous dynamics is removed from the models resulting in unique, Markovian (memoryless), (quasi)-equilibrium solutions. This procedure is illustrated in Figure 1 where the complicated 'human filter' that lies between the new information and the aggregate variables (such as price) does not alter its Brownian nature. This then justifies the use of SDEs upon aggregate variables directly.

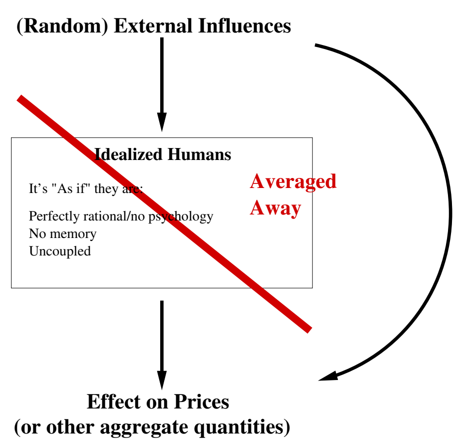

Fig. 1

However, the reality is far more complicated, as shown in Figure 2. Important human characteristics such as psychology, memory, systemic cognitive or emotional biases, adaptive heuristics, group influences and perverse incentives will be present, as well as various possible positive feedbacks caused by endogenous dynamics or interactions with the aggregate variables.

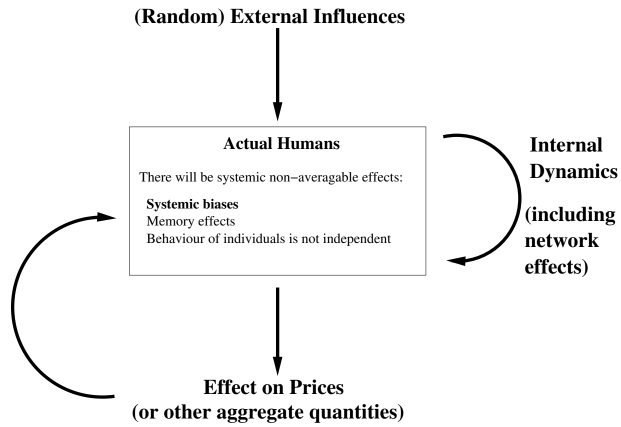

Fig. 2

In an attempt to incorporate some of these effects, many Heterogeneous Agent Models (HAMs) have been developed [6] that simulate agents directly rather than operate on the aggregate variables. These have demonstrated that it is relatively easy to generate aggregate output data, such as the price of a traded asset, that approximate reality better than the standard averaging-type models. In particular the seemingly universal ‘stylized facts’ [11, 2] of financial markets such as *heteroskedasticity* (volatility clustering) and *leptokurtosis* (fat-tailed price-return distributions resulting from booms-and-busts) have been frequently reproduced. However, the effects of such research upon mainstream modeling have been minimal perhaps, in part, because some HAMs require fine tuning of important parameters, others are too complicated to analyze, and the plethora of different HAMs means that many are mutually incompatible.

The purpose of this paper is rather different from other studies involving HAMs. We are not proposing a stand-alone asset pricing model to replace extant ones (although it could indeed be used that way). Rather we are proposing an entire framework in which the stability of equilibrium models can be tested in the presence of a wide class of non-standard perturbations that weaken various aspects of the efficiency and rationality assumptions — in particular those related to a), b) and c) above.

In this paper we focus upon the effects of herding as it is an easily understood phenomenon that has multiple causes (rational, psychological or due to perverse incentives) and is an obvious source of lack of independence between agents' actions. We start by introducing a simplified version of the modeling framework introduced in [10, 9] that can also be described as a particle system in two dimensions (Figure 3). A web-based interactive simulation of the model can be found at http://math.gmu.edu/∼harbir/market.html . It provides useful intuition as to how endogenous multi-year boom-and-bust dynamics naturally arise from the competition between equilibrating and dis-equilibrating forces that are usually only considered important over much shorter timescales.

2 A stochastic particle system with re-injection and switching

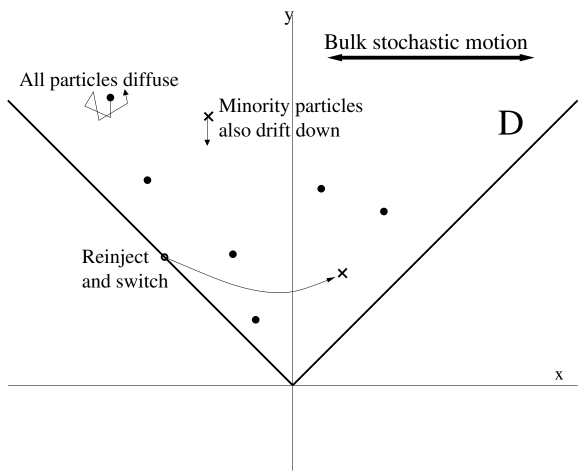

Fig. 3 The M signed particles are subject to a horizontal stochastic forcing and they also diffuse independently. Minority particles also drift downwards at a rate proportional to the imbalance. When a particle hits the boundary it is re-injected with the opposite sign and a kick is added to the bulk forcing that can trigger a cascade (see text).

We define the open set $D \subset \Re^2$ by $D = \{(x,y) : -y < x < y,\ y > 0\}$. There are M signed particles (with states $+1$ or -1) that move within D subject to four different motions. Firstly there is a bulk Brownian forcing B_t in the x-direction that acts upon every particle. Secondly, each particle has its own independent two-dimensional diffusion process. Thirdly, for agents *in the minority state only*, there is a downward (negative y-direction) drift that is proportional to the imbalance.

Finally, when a particle hits the boundary ∂D it is re-injected into D *with the opposite sign* according to some predefined probability measure. When this happens, the position of the other particles is kicked in the x-direction by a (small) amount $\pm\frac{2\kappa}{M}$, $\kappa > 0$, where the kick is positive if the switching particle goes from the -1 state to $+1$ and negative if the switch is in the opposite direction. Note that the particles do not interact locally or collide with one another.

2.1 Financial market interpretation

We take as our starting point the standard geometric Brownian motion (gBm) model of an asset price p_t at time t with $p_0 = 1$. It is more convenient to use the log-price $r_t = \ln p_t$ which for constant drift a and volatility b is given by the solution $r_t = at + bB_t$ to the SDE

$$dr_t = a\,dt + b\,dB_t. \tag{1}$$

Note that the solution r_t depends only upon the value of the exogenous Brownian process B_t *at time* t and not upon $\{B_s\}_{s=0}^{t}$. This seemingly trivial observation implies that r_t is Markovian and consistent with various notions of market efficiency. Thus gBm can be considered a paradigm for economic and financial models in which the aggregate variables are assumed to be in a quasi-equilibrium reacting instantaneously and reversibly to new information.

The model involves two types of agent and a separation of timescales. 'Fast' agents react near instantaneously to the arrival of new information B_t. Their effect upon the asset price is close to the standard models and they will not be modeled directly. However, we posit the existence of M 'slow' agents who are primarily motivated by price changes rather than new information and act over much longer timescales (weeks or months). At time t the i^{th} slow agent is either in state $s_i(t) = +1$ (owning the asset) or $s_i(t) = -1$ (not owning the asset) and the *sentiment* $\sigma(t) \in [-1,1]$ is defined as $\sigma(t) = \frac{1}{M}\sum_{i=1}^{M} s_i(t)$. The i^{th} slow agent is deemed to have an evolving strategy that at time t consists of an open interval $(L_i(t), U_i(t))$ containing the current log-price r_t (see Figure 4). The i^{th} agent switches state whenever the price crosses either threshold, ie $r_t = L_i(t)$ or $U_i(t)$, and a new strategy interval is generated straddling the current price. Note that slow agents wishing to trade do not need to be matched with a trading partner — it is assumed that the fast agents provide sufficient liquidity.

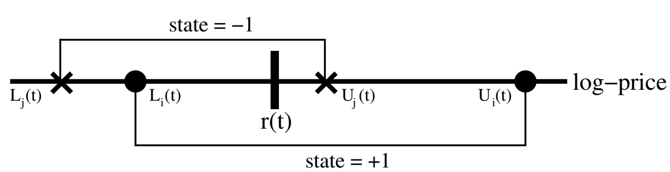

Fig. 4 A representation of the model showing two agents in opposite states at time t. Agent i is in the $+1$ state and is represented by the two circles at $L_i(t)$ and $U_i(t)$ while agent j is in the -1 state and is represented by the two crosses.

We assume in addition that each threshold for every slow agent has its own independent diffusion with rate α_i (corresponding to slow agents' independently evolving strategies) and those in the minority (ie. whose state differs from $|\sigma|$) also have their lower and upper thresholds drift inwards each at a rate $C_i|\sigma|$, $C_i \geq 0$.

These *herding constants* C_i are crucial as they provide the only (global) coupling between agents. The inward drift of the minority agents' strategies makes them more likely to switch to join the majority — they are being pressured/squeezed out of their minority view. Herding, and other mimetic effects, appear to be a common feature of financial and economic systems. Some causes are irrationally human while others may be rational responses by, for example, fund managers not wishing to deviate too far from the majority opinion and thereby risk severely under-performing their average peer performance. The reader is directed to [9] for a more detailed discussion of these and other modeling issues.

Finally, changes in the sentiment σ feed back into the asset price so that gBm (1) is replaced with

$$dr_t = a\,dt + b\,dB_t + \kappa\Delta\sigma \tag{2}$$

where $\kappa > 0$ and the ratio κ/b is a measure of the relative impact upon r_t of exogenous information versus endogenous dynamics. Without loss of generality we let $a = 0$ and $b = 1$ by setting the risk-free interest rate to zero and rescaling time.

One does not need to assume that all the slow agents are of equal size, have equal strategy-diffusion, and equal herding propensities. But if one does set $\alpha_i = \alpha$ and $C_i = C\ \forall i$ then the particle system above is obtained by representing the i^{th} agent, not as an interval on $\Re$, but as a point in $D \subset \Re^2$ with position $(x_i, y_i) = (\frac{U_i+L_i}{2} - r_t, \frac{U_i-L_i}{2})$. To make the correspondence explicit: the bulk stochastic motion is due to the exogenous information stream, the individual diffusions are caused by strategy-shifting of the slow agents; the downward drift of minority agents is due to herding; the re-injection and switching are the agents changing investment position; and the kicks that occur at switches are due to the change in sentiment affecting the asset price via the linear supply/demand price assumption.

2.2 Limiting values of the parameters

There are different parameter limits that are of interest.

1) $\underline{M \to \infty}$ In the continuum limit the particles are replaced by a pair of evolving density functions $\rho^+(x,y,t)$ and $\rho^-(x,y,t)$ representing the density of each agent state on D — such a mesoscopic Fokker-Planck description of a related, but simpler, market model can be found in [5]. The presence of nonstandard boundary conditions, global coupling, and bulk stochastic motion present formidable analytic challenges for even the most basic questions of existence and uniqueness of solutions. However, numerical simulations strongly suggest that, minor discretization effects aside, the behaviour of the system is independent of M for $M > 1000$.

2) $\underline{B_t \to 0}$ As the external information stream is reduced the system settles into a state where σ is close to either ± 1. Therefore this potentially useful simplification is not available to us.

3) $\underline{\alpha \to 0 \text{ or } \infty}$ In the limit $\alpha \to 0$ the particles do not diffuse ie. the agents do not alter their thresholds between trades/switches. This case was examined in [3] and the lack of diffusion does not significantly change the boom-bust behaviour shown below. On the other hand, for $\alpha \gg \max(1, C)$ the diffusion dominates both

the exogenous forcing and the herding/drifting and equilibrium-type dynamics is re-established. This case is unlikely in practice since slow agents will alter their strategies more slowly than changes in the price of the asset.
4) $\underline{C \to 0}$ This limit motivates the next Section. When $C = 0$ the particles are uncoupled and if the system is started with approximately equal distributions of ± 1 states then σ remains close to 0. Thus (2) reduces to (1) and the particle system becomes a standard equilibrium model — agents have differing expectations about the future which causes them to trade but on average the price remains 'correct'. In Section 3 we shall observe that endogenous dynamics arise as C is increased from 0 and the equilibrium gBm solution loses stability in the presence of even small amounts of herding.
5) $\underline{\kappa \to 0}$ For $\kappa > 0$ even one agent switching can cause an avalanche of similar switches, especially when the system is highly one-sided with $|\sigma|$ close to 1. When $\kappa = 0$ the particles no longer provides kicks (or affect the price) when they switch although they are still coupled via $C > 0$. The sentiment σ can still drift between -1 and $+1$ over long timescales but switching avalanches and large, sudden, price changes do not occur.

3 Parameter estimation, numerical simulations and the instability of Geometric Brownian pricing

In all the simulations below we use $M = 10000$ and discretize using a timestep $h = 0.000004$ which corresponds to approximately 1/10 of a trading day if one assumes a daily standard deviation in prices of $\approx 0.6\%$ due to new information. The price changes of 10 consecutive timesteps are then summed to give daily price return data making the difference between synchronous vs asynchronous updating relatively unimportant.

We choose $\alpha = 0.2$ so that slow agents' strategies diffuse less strongly than the price does. A conservative choice of $\kappa = 0.2$ means that the difference in price between neutral ($\sigma = 0$) and polarized markets $\sigma = \pm 1$ is, from (2), $\exp(0.2) \approx 22\%$.

After switching, an agent's thresholds are chosen randomly from a Uniform distribution to be within 5% and 25% higher and lower than the current price. This allows us to estimate C by supposing that in a moderately polarized market with $|\sigma| = 0.5$ a typical minority agent (outnumbered 3–1) would switch due to herding pressure after approximately 80 trading days (or 3 months, a typical reporting period for investment performance)[14]. The calculation $80C|\sigma| = |\ln(0.85)|/0.00004$ gives $C \approx 100$. Finally, we note that no fine-tuning of the parameters is required for the observations below.

Figure 5 shows the results of a typical simulation, started close to equilibrium with agents' states equally mixed and run for 40 years. The difference in price history between the above parameters and the equilibrium gBm solution is shown in the top left. The sudden market reversals and over-reactions can be seen more clearly

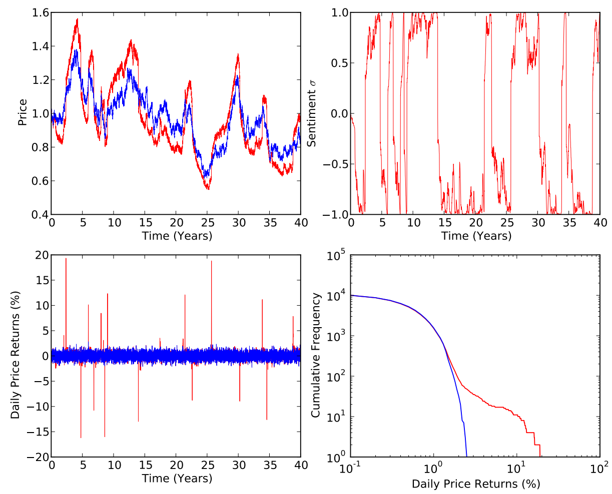

Fig. 5 In each picture, red corresponds to outputs for the parameters used in the text while blue represents outputs for the gBm pricing model (1) with the same exogenous information stream B_s. The top left figure shows the prices $p(t)$; top right is sentiment vs time; bottom left plots the daily price changes; and bottom right shows the cumulative log-log plot of daily price changes that exceed a given percentage.

in the top right plot where the market sentiment undergoes sudden shifts due to switching cascades. These result in price returns (bottom left) that could quite easily bankrupt anyone using excessive financial leverage and gBm as an asset pricing model! Finally in the bottom right the number of days on which the magnitude of the price change exceeds a given percentage is plotted on log-log axes. It should be emphasized that this is a simplified version of the market model in [9] and an extra parameter that improves the statistical agreement with real price data (by inducing volatility clustering) has been ignored.

To conclude we examine the stability of the equilibrium gBm solution using the herding level C as a bifurcation parameter. In order to quantify the level of disequilibrium in the system we record the maximum value of $|\sigma|$ ignoring the first 10 years of the simulation (to remove any possible transient effects caused by the initial conditions) and average over 20 runs each for values of $0 \leq C \leq 40$. All the other parameters and the initial conditions are kept unchanged.

The results in Figure 6 show that even for values of C as low as 20 the deviations from the equilibrium solution are close to being as large as the system will allow with $|\sigma|$ usually getting close to ± 1 at some point during the simulations. To reiterate, this occurs at a herding strength C which is a factor of 5 lower than the value of $C = 100$ estimated above for real markets! It should also be noted that there are other significant phenomena that have not been included, such as new investors and money entering the asset market after a bubble has started, and localized interactions between certain subsets of agents. These can be included in the model by allowing κ to vary (increasing at times of high market sentiment for example) and, as expected, they cause the equilibrium solution to destabilize even more rapidly.

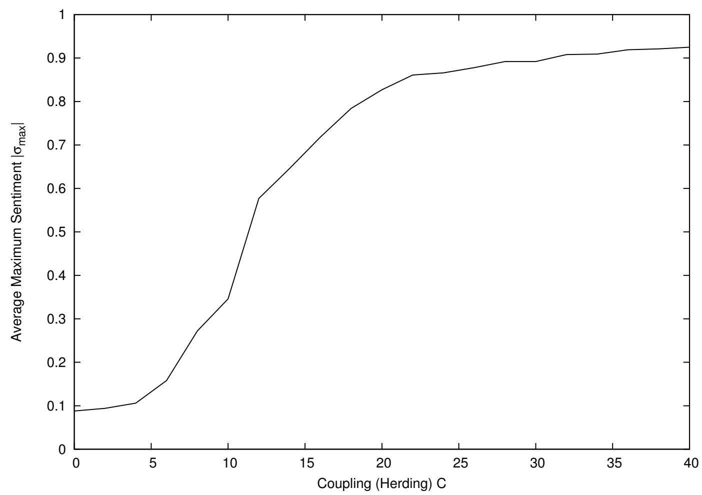

Fig. 6 A measure of disequilibrium $|\sigma|_{\max}$ averaged over 20 runs as the herding parameter C changes.

4 Conclusions

Financial and economic systems are subject to many different kinds of interdependence between agents and potential positive feedbacks. However, even those mainstream models that attempt to quantify such effects [1, 14] assume that the result will merely be a shift of the equilibria to nearby values without qualitatively changing the nature of the system. We have demonstrated that at least one such form of coupling (incremental herding pressure) results in the loss of stability of the equilibrium. Furthermore the new dynamics occurs at realistic parameters and is clearly recognizable as 'boom-and-bust'. It is characterized by multi-year periods of low-level endogenous activity (long enough, certainly, to convince equilibrium-believers that the system is indeed in an equilibrium with slowly varying parameters) followed by large, sudden, reversals involving cascades of switching agents triggered by price changes.

A similar model was studied in [8] where momentum-traders replaced the slow agents introduced above. The results replicated the simulations above in the sense that the equilibrium solution was replaced with multi-year boom-and-bust dynamics but with the added benefit that analytic solutions can be derived, even when agents are considered as nodes on an arbitrary network rather than being coupled globally.

The model presented here is compatible with existing (non-mathematized) critiques of equilibrium theory by Minsky and Soros [12, 15]. Furthermore, work on related models to appear elsewhere shows that positive feedbacks can result in similar non-equilibrium dynamics in more general micro- and macro-economic situations.

Acknowledgements The author thanks Michael Grinfeld, Dmitri Rachinskii and Rod Cross for numerous enlightening conversations and Julian Todd for writing the browser-accessible simulation of the particle system.

References

1. G. Akerlof and J. Yellen. Can small deviations from rationality make significant differences to economic equilibria? *The American Economic Review*, 75(4):708–720, 1985.
2. R. Cont. Empirical properties of asset returns: stylized facts and statistical issues. *Quantitive Finance*, 1:223–236, 2001.
3. R. Cross, M. Grinfeld, H. Lamba, and T. Seaman. A threshold model of investor psychology. *Phys. A*, 354:463–478, 2005.
4. E.F. Fama. The behavior of stock market prices. *Journal of Business*, 38:34–105, 1965.
5. M. Grinfeld, H. Lamba, and R. Cross. A mesoscopic market model with hysterestic agents. *Discr. Cont. Dyn. Sys. B*, 18:403–415, 2013.
6. Cars H. Hommes. volume 2 of *Handbook of Computational Economics*, pages 1109 – 1186. Elsevier, 2006.
7. A. Kirman. Who or what does the representative agent represent? *Journal of Economic Perspectives*, 6:117–136, 1992.
8. P. Krejčí, S. Melnik, H. Lamba, and D. Rachinskii. Analytical solution for a class of network dynamics with mechanical and financial applications. *Phys. Rev E*, 90:032822, 2014.
9. H. Lamba. A queueing theory description of fat-tailed price returns in imperfect financial markets. *Eur. Phys. J. B*, 77:297–304, 2010.
10. H. Lamba and T. Seaman. Rational expectations, psychology and learning via moving thresholds. *Phys. A*, 387:3904–3909, 2008.
11. R. Mantegna and H. Stanley. *An Introduction to Econophysics*. CUP, 2000.
12. H.P. Minsky. The financial instability hypothesis. *The Jerome Levy Institute Working Paper*, 74, 1992.
13. J.A. Muth. Rational expectations and the theory of price movements. *Econometrica*, 6, 1961.
14. D. Scharfstein and J. Stein. Herd behavior and investment. *The American Economic Review*, 80(3):465–479, 1990.
15. G. Soros. *The alchemy of finance*. John Wiley & Sons Inc, 1987.