$$\int \sin^m x \cos^n x \, dx$$

ii) If m = even & n odd, $n = 2l+1$
we do the similar thing

$$\int \sin^m x \cos^{2l+1} x \, dx$$

$$= \int \sin^m x \cos^{2l} x \cdot \cos x \, dx$$

$$= \int \sin^m x (1-\sin^2 x)^l \cdot \cos x \, dx$$

$\Rightarrow$ $u = \sin x$
$du = \cos x \, dx$

$$= \int u^m (1-u^2)^l \, du \quad \ldots$$

ex $\int \cos^5 x \, dx = \int \cos^4 x \cdot \cos x \, dx$

$$= \int (1-\sin^2 x)^2 \cos x \, dx \quad \begin{pmatrix} u = \sin x \\ du = \cos x \, dx \end{pmatrix}$$

$$= \int (1-u^2)^2 \, du$$

$$= \int (1 - 2u^2 + u^4)\,du$$

$$= u - \frac{2}{3}u^3 + \frac{1}{5}u^5$$

$$= \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.$$

(iii) If both m & n are even, this is the hardest case.

Say $m = 2k$, $n = 2\ell$

$$\int \sin^m x \cos^n x\,dx$$

$$= \int \sin^{2k} x \cos^{2\ell} x\,dx$$

$$= \int (\sin^2 x)^k (\cos^2 x)^\ell\,dx$$

$$= \int \left(\frac{1 - \cos 2x}{2}\right)^k \left(\frac{1 + \cos 2x}{2}\right)^\ell dx$$

& multiply out.. !

ex $\int \sin^2 x \cos^4 x \, dx$

$$= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= \int \frac{1}{2}(1-\cos 2x)\cdot\frac{1}{4}(1+\cos 2x)^2 dx$$

$$= \frac{1}{8}\int (1-\cos 2x)(1+\cos 2x)^2 dx$$

$$= \frac{1}{8}\int (1-\cos^2 2x)(1+\cos 2x) dx$$

$$= \frac{1}{8}\int (1+\cos 2x - \cos^2 2x - \cos^3 2x) dx$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int \cos^2 2x \, dx - \int \cos^3 2x \, dx\right]$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int \left(\frac{1+\cos 4x}{2}\right) dx - \int \cos^3 2x \, dx\right]$$

$$= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x + \left(\frac{1}{2}x + \frac{1}{8}\sin 4x\right) - \int \cos^2 2x \cdot \cos 2x \, dx \right]$$

$u = \sin 2x$, $du = 2\cos 2x \, dx$

$du/2 = \cos 2x \, dx$

$$\int (1 - \sin^2 2x) \cdot \cos 2x \, dx$$

$$= \int (1-u^2) \frac{du}{2} = \frac{1}{2}\int (1-u^2)\, du$$

$$= \frac{1}{2}\left[u - \frac{1}{3}u^3\right]$$

$$= \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right)$$

So $I =$

$$\frac{1}{8}\left[x + \frac{1}{2}\sin 2x + \frac{1}{2}x - \frac{1}{8}\sin 4x - \left(\frac{1}{2}\sin 2x - \frac{1}{6}\sin^3 2x\right)\right]$$

$$= \frac{1}{8}\left[x + \cancel{\frac{1}{2}\sin 2x} - \frac{1}{2}x - \frac{1}{8}\sin 4x - \cancel{\frac{1}{2}\sin 2x} + \frac{1}{6}\sin^3 2x\right]$$

$$\frac{1}{8}\left[\frac{1}{2}x - \frac{1}{8}\sin 4x + \frac{1}{6}\sin^3 2x\right]$$

$$\frac{1}{16}\left[x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right] + C.$$

- What about $\tan^n x$?
- Here $\frac{d}{dx}\tan x = \sec^2 x = \tan^2 x + 1$

$$\Rightarrow \int \tan^n x\,dx = \int \tan^{n-2} x \cdot \tan^2 x\,dx$$

$$= \int \tan^{n-2} x\,(\tan^2 x + 1)\,dx - \int \tan^{n-2} x\,dx$$

$u = \tan x$

$du = (\tan^2 x + 1)\,dx$

$$= \int u^{n-2} \cdot du - \int \tan^{n-2} x\,dx$$

$$= \frac{1}{n-1} u^{n-1} - \int \tan^{n-2} x \, dx$$

<u>So:</u>

$$\boxed{\int \tan^n x \, dx = \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \, dx.}$$

ex $\int \tan^5 x \, dx \quad (n = 5!)$

$$= \frac{1}{4} \tan^4 x - \int \tan^3 x \, dx \qquad n=3$$

$$= \frac{1}{4} \tan^4 x - \left[\frac{1}{2} \tan^2 x - \int \tan x \, dx \right]$$

$$= \frac{1}{4} \tan^4 x - \frac{1}{2} \tan^2 x + \int \tan x \, dx$$

$$\int \frac{\sin x}{\cos x} dx \qquad \begin{array}{l} u = \cos x \\ du - \sin x \, dx \end{array}$$

$$= \int -\frac{du}{u} = -\log|u|$$

$$= -\log|\cos x|$$

$$= \log|\sec x|$$

so: $\int \tan^5 x \, dx$

$$= \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \log|\sec x| + C.$$

Similarly we can for $\sec^n x$ obtain an equation (as for $\int e^x \sin x \, dx$)

ex $\int \sec^3 x \, dx$

$$= \int \underset{N}{\sec x} \cdot \underset{D}{\sec^2 x} \, dx$$

$$= \underset{N}{\sec x} \cdot \underset{N}{\tan x} - \int \underset{D}{(\sec x \cdot \tan x)} \underset{N}{\tan x} \, dx$$

$$= \sec x \tan x - \int \sec x \cdot \tan^2 x \, dx$$

$$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$$

$$\sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$

$$= \sec x \tan x - \int \sec^3 x \, dx + \log|\sec x + \tan x|$$

<u>So</u>:

$$2\int \sec^3 x \, dx = \sec x \tan x + \log|\sec x + \tan x|$$

or

$$\int \sec^3 x \, dx = \frac{1}{2}\left(\sec x \tan x + \log|\sec x + \tan x|\right) + C.$$

– This can also be used to obtain a <u>reduction formula</u> for

$$\int \sec^n x \, dx \quad \ldots \quad !$$

Product of cosines & sines :

- To compute integrals of the form

$$\int \sin\alpha x \cdot \sin\beta x \, dx,$$

$$\int \sin\alpha x \cdot \cos\beta x \, dx,$$

$$\int \cos\alpha x \cdot \cos\beta x \, dx,$$

then use :

$$\sin\alpha x \sin\beta x = \frac{1}{2}\left(\cos(\alpha-\beta)x - \cos(\alpha+\beta)x\right)$$
$$\sin\alpha x \cos\beta x = \frac{1}{2}\left(\sin(\alpha-\beta)x + \sin(\alpha+\beta)x\right)$$
$$\cos\alpha x \cos\beta x = \frac{1}{2}\left(\cos(\alpha-\beta)x + \cos(\alpha+\beta)x\right)$$

ex $\int \sin 7x \cdot \sin 11x \, dx$

$$= \int \frac{1}{2}\left(\cos(7-11)x - \cos(7+11)x\right) dx$$

$$= \frac{1}{2}\int \cos(-4x)\,dx - \frac{1}{2}\int \cos(18x)\,dx$$

$$= \frac{1}{2}\int \cos 4x\,dx - \frac{1}{2}\int \cos 18x\,dx$$

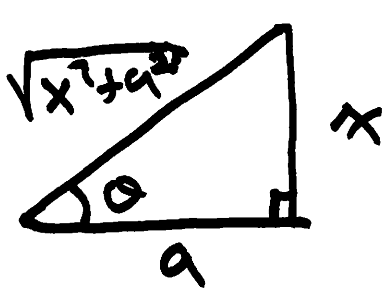
$$= \frac{1}{2}\cdot\frac{1}{4}\sin 4x - \frac{1}{2}\cdot\frac{1}{18}\sin 18x$$

$$= \frac{1}{8}\sin 4x - \frac{1}{36}\sin 18x + C.$$

Trig substitutions:

- Trigonometric substitutions (TS) enable us to get rid of $\sqrt{x^2+a^2}$, $\sqrt{x^2-a^2}$, $\sqrt{a^2-x^2}$ when they appear in $f(x)$ to be integrated.
- There are three basic substitutions:

(1) For $\sqrt{x^2+a^2}$:



$x = a\tan\theta$

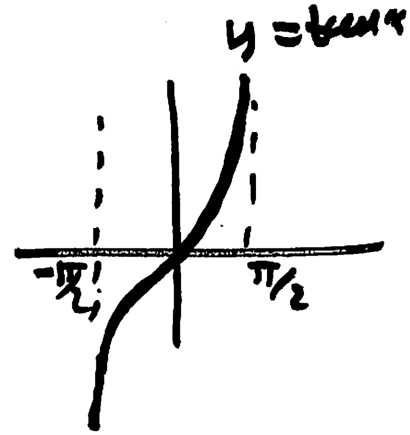
$dx = a\sec^2\theta\,d\theta$

so $\sqrt{x^2+a^2} = \sqrt{a^2\tan^2\theta + a^2}$

$= \sqrt{a^2(\tan^2\theta + 1)}$

$= \sqrt{a^2\sec^2\theta} = a|\sec\theta|$

$= a\sec\theta$

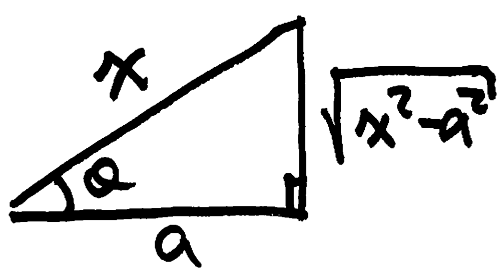


for $-\pi/2 < \theta < \pi/2$

& so

$\theta = \arctan\left(\frac{x}{a}\right)$

② For $\sqrt{x^2-a^2}$:



$\frac{a}{x} = \cos\theta$

$x = \frac{a}{\cos\theta} = a\sec\theta$

$dx = a\sec\theta\tan\theta\, d\theta$

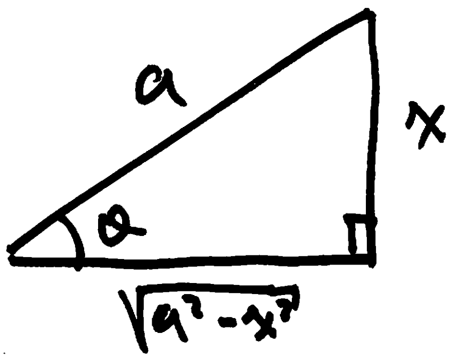
so have

$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta - a^2}$

$= \sqrt{a^2(\sec^2\theta - 1)}$

$= a\tan\theta \ldots$

③ For $\sqrt{a^2-x^2}$:

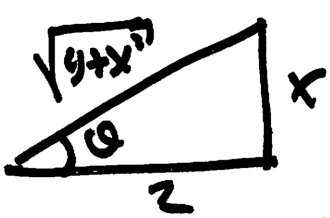


$$\frac{x}{a} = \sin\theta$$
$$x = a\sin\theta$$
$$dx = a\cos\theta \, d\theta$$

$$\Rightarrow \sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta}$$
$$= \sqrt{a^2(1-\sin^2\theta)} = a\cos\theta$$
$$-\pi/2 < x < \pi/2$$

ex $\displaystyle\int \frac{dx}{\sqrt{4+x^2}} = I$

$$x = 2\tan\theta$$
$$dx = 2\sec^2\theta \, d\theta$$



so $\displaystyle I = \int \frac{2\sec^2\theta \, d\theta}{2\sec\theta} = \int \sec\theta \, d\theta$

look ② △ $= \log|\sec\theta + \tan\theta|$

from Δ

$$= \log\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C$$

$$= \log\left|\tfrac{1}{2}\left(\sqrt{4+x^2} + x\right)\right| + C$$

$$= \log\tfrac{1}{2} + \log\left|\sqrt{4+x^2} + x\right| + C$$

$$= \log\left|\sqrt{4+x^2} + x\right| + C$$

$$= \log\left(\sqrt{4+x^2} + x\right) + C.$$

$$\left(\sqrt{4+x^2} \geq |x| \geq x \,!\right)$$

— What if you dont remember $\int \sec\theta \, d\theta$?

$$\int \sec\theta \, d\theta = \int \frac{d\theta}{\cos\theta} = \int \frac{\cos\theta \, d\theta}{\cos^2\theta}$$

$$= \int \frac{\cos\theta \, d\theta}{1-\sin^2\theta} \qquad \begin{array}{l} u = \sin\theta \\ du = \cos\theta \, d\theta \end{array}$$

$$= \int \frac{du}{1-u^2}$$

$$= \int \frac{-du}{u^2-1} = \int \left(\frac{-1}{u^2-1}\right) du$$

$$= \int \left(\frac{-1}{(u-1)(u+1)}\right) du$$

$$= \int \left(\frac{-1/2}{u-1} + \frac{-1/-2}{u+1}\right) du$$

$$= \int \left(\frac{1/2}{1+u} + \frac{1/2}{1-u}\right) du$$

$$= \frac{1}{2}\log|1+u| - \frac{1}{2}\log|1-u|$$

$$= \frac{1}{2}\log\left|\frac{1+u}{1-u}\right|$$

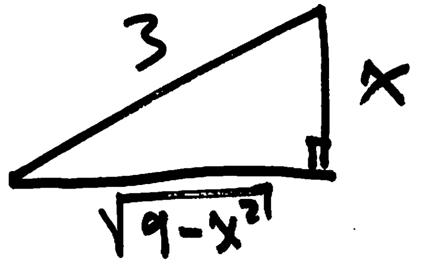
$$= \log\sqrt{\frac{1+u}{1-u}}$$

$$= \log\sqrt{\frac{1+\sin\alpha}{1-\sin\alpha}} + C.$$

ex $\displaystyle\int \frac{x^3\,dx}{\sqrt{9-x^2}} = I$

$\frac{x}{3} = \sin\theta$

$dx = 3\cos\theta\, d\theta$



$$I = \int \frac{\sin^3\theta \cdot 27}{3\cdot\cos\theta}\cdot 3\cos\theta\, d\theta$$

$$= 27\int \frac{\sin^3\theta\cdot\cos\theta}{\cos\theta}\, d\theta$$

$$= 27\int \sin^3\theta\, d\theta$$

$$= 27\int (1-\cos^2\theta)\sin\theta\, d\theta$$

$u = \cos\theta$

$du = -\sin\theta\, d\theta$

$$= -27\int (1-u^2)\cdot du$$

$$= -27\left(u - \tfrac{1}{3}u^3\right)$$

$$= -27\left(\cos\theta - \tfrac{1}{3}\cos^3\theta\right)$$

$$= 9\cos^3\theta - 27\cos\theta$$

$$= 9\left(\frac{\sqrt{9-x^2}}{3}\right)^3 - 27\left(\frac{\sqrt{9-x^2}}{3}\right)$$

$$= \frac{\sqrt{9-x^2}}{3}\left(9\cdot\left(\frac{\sqrt{9-x^2}}{3}\right)^2 - 27\right)$$

$$= \frac{\sqrt{9-x^2}}{3}\left((9-x^2) - 27\right)$$

$$= \frac{\sqrt{9-x^2}}{3}\left(-x^2 - 18\right)$$

$$= -\frac{\sqrt{9-x^2}}{3}(x^2+18) + C.$$