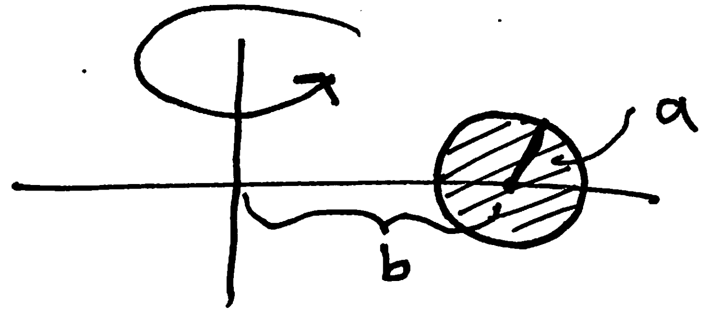
(ex) By Pappus: the volume of a torus is:



$$V = 2\pi \cdot b \cdot A$$
$$= 2\pi b\,(\pi a^2) = \underline{\underline{2\pi^2 b a^2}}$$

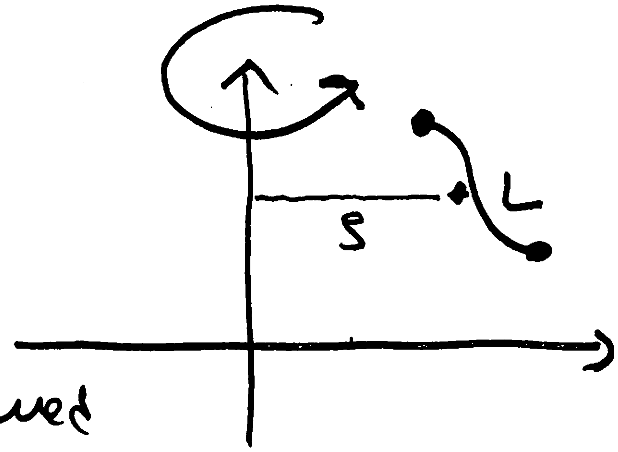
(ex) Volume of a sphere (using Pappus)

$$CM = (\bar{x}, \bar{y}) = \left(0, \frac{4R}{3\pi}\right)$$
$$\Rightarrow \rho = \frac{4R}{3\pi}$$

& so $V = 2\pi\rho \cdot A$

$$= 2\pi\left(\frac{4R}{3\pi}\right)\left(\frac{1}{2}\pi R^2\right)$$
$$= \underline{\underline{\frac{4}{3}\pi R^3}}.$$

Thm 2

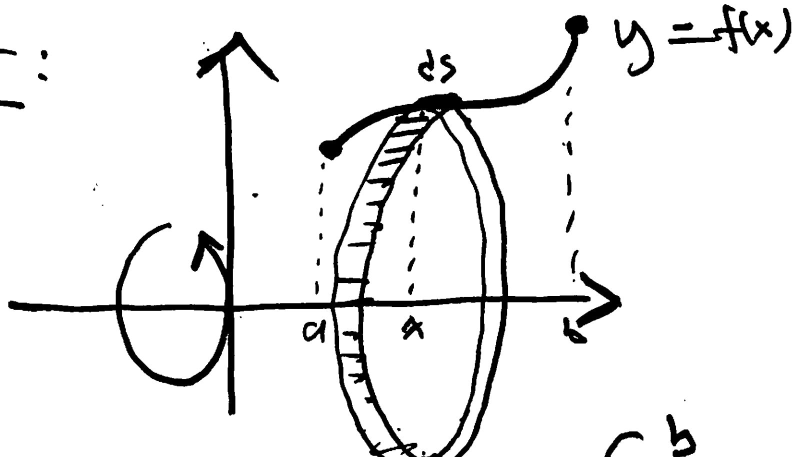


<u>Surface area:</u>

of this region formed by rotation is:

$$\boxed{A = 2\pi\rho \cdot L}$$

<u>Pf:</u>



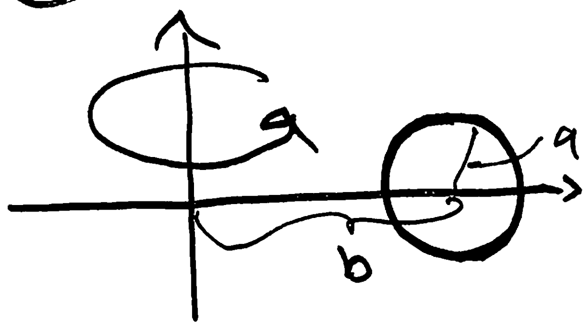
We know: $A = \displaystyle\int_a^b 2\pi\, ds \cdot y = 2\pi \int_a^b y\, ds$

Also: $CM = (\bar{x}, \bar{y})$

$$\bar{y} = \frac{\int_a^b y\, ds}{\int_a^b ds} = \frac{\int_a^b y\, ds}{L} \quad \text{so} \quad \int_a^b y\, ds = \bar{y} \cdot L = \rho \cdot L$$

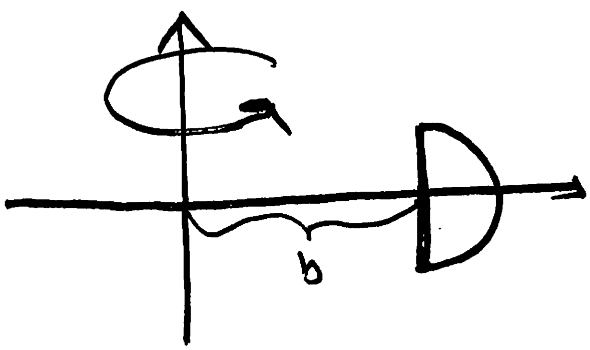
$$\Rightarrow A = 2\pi \int_a^b y\, ds = \underline{2\pi\rho L} \quad \blacksquare$$

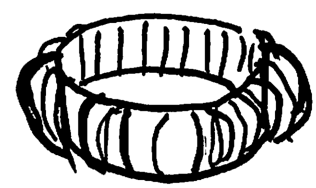
ex Surface area of a torus:



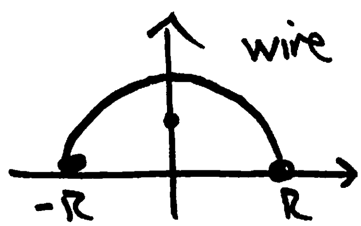
$$A = 2\pi \cdot b \cdot L$$
$$= 2\pi b \cdot (2\pi a)$$
$$= \underline{\underline{4\pi^2 ab}}$$

ex Surface are of

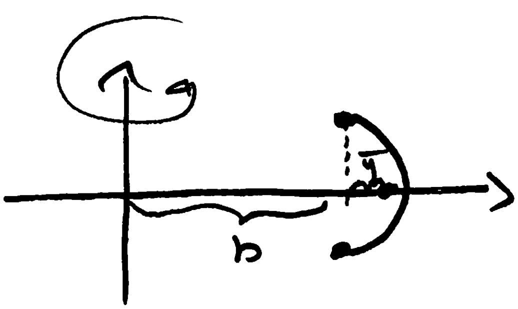




We know:



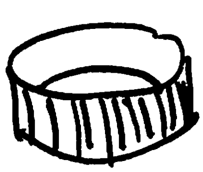
$$CM = (\bar{x}, \bar{y})$$
$$= (\bar{0}, \tfrac{2}{\pi} R)$$



$$\rho = b + \tfrac{2}{\pi} a$$

So $A_1 = 2\pi \rho \cdot L = 2\pi\left(b + \tfrac{2}{\pi} a\right) \cdot \left(\tfrac{1}{2} 2\pi a\right)$

$$= 2\pi^2 a \left(b + \tfrac{2}{\pi} a\right)$$



$$A_2 = 2\pi b \cdot 2a = 4\pi ab$$

$$A_{total} = A_1 + A_2 = 2\pi^2 a \left(b + \tfrac{2}{\pi} a\right) + 4\pi a b$$
$$\underline{\underline{2\pi\left(\pi ab + 2a^2 + 2ab\right)}}.$$

$$= 2\pi a(\pi b + 2a + 2b)$$
$$= 2\pi a(2a + (2+\pi)b)$$

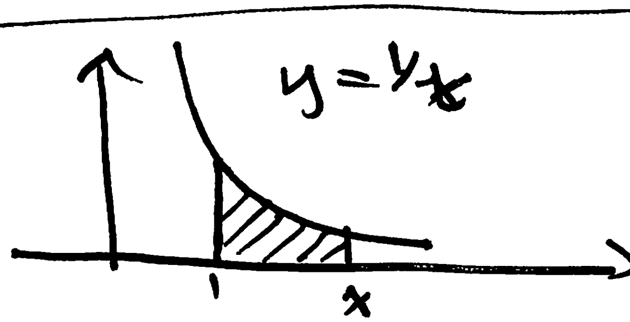
Integrals & transcedental functions

- Defining the logarithm:

So far we have been using log & exp without proper definition (in an ad hoc way ....)

- The precise definition of the natural logarithm is

$$\boxed{\log x := \int_1^x \frac{dt}{t}, \quad x \geq 0}$$



$\log x$ = area between $y = 1/t$ & t-axis from 1 to x

— With this definition we can deduce all the properties we so far taken for granted.

(ex) $$\log(xy) = \int_1^{xy} \frac{dt}{t} = \int_1^{x} \frac{dt}{t} + \int_x^{xy} \frac{dt}{t}$$

$$= \log x + \int_x^{xy} \frac{dt}{dt}$$

$$x \le t \le xy \iff 1 \le t/x \le y$$

substitution: $u = t/x \quad du = dt/x$

as t ranges from x to xy

then u — — 1 — y

so $$\int_x^{xy} \frac{dt}{t} = \int_1^{y} \frac{x\,du}{xu} = \int_1^{y} \frac{du}{u} = \log y$$

$$\implies \log(xy) = \log x + \log y \quad \blacksquare$$

NB!

- If $x>0$ then $\frac{d}{dx}\log|x| = \frac{d}{dx}\log x = \frac{1}{x}$
- If $x<0$ then $\frac{d}{dx}\log|x| = \frac{d}{dx}\log(-x)$

$$= \frac{1}{-x}\cdot(-1) = \frac{1}{x}$$

(Chain Rule)

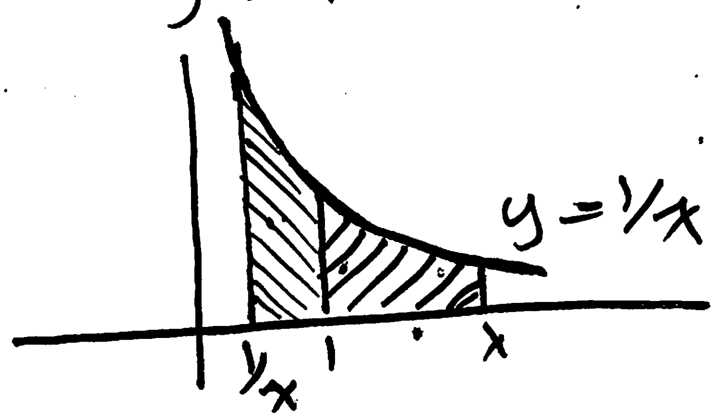
Regardless of sign:

$$\frac{d}{dx}\log|x| = \frac{1}{x} \quad (x \neq 0)$$

or $$\boxed{\int \frac{dx}{x} = \log|x| + C}$$

— All other familiar properties follow from its definition of log

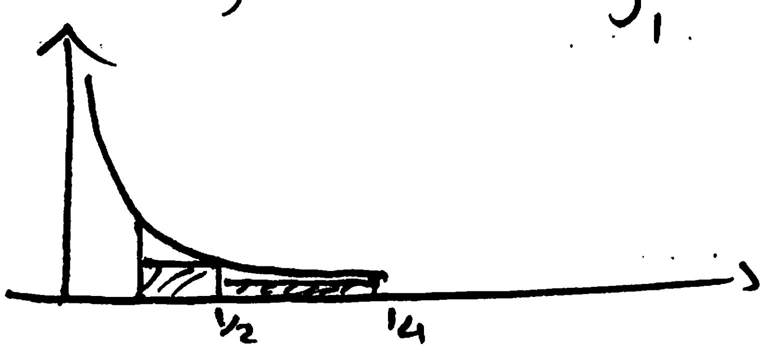
(e.g. $\log\left(\frac{1}{x}\right) = -\log x$ ..)



ex $\lim_{x \to \infty} \log x = \infty$:

For k an integer let $x = 2^k$ so

$$\log x = \log(2^k) = \int_1^{2^k} \frac{dt}{t}$$



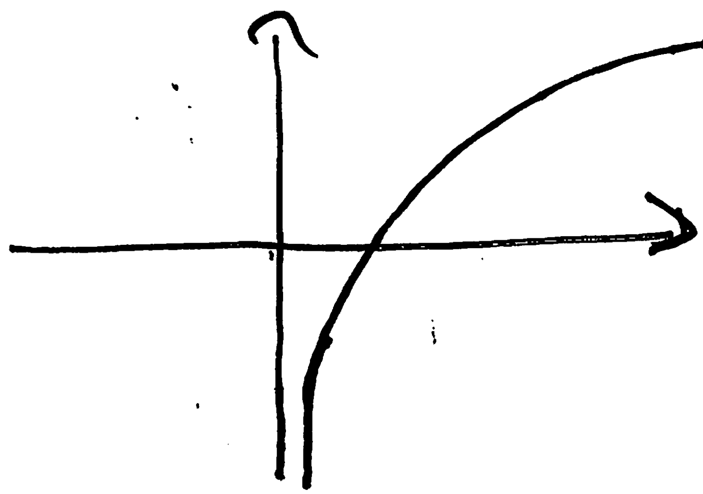
so $\int_1^{2^k} \frac{dt}{t} > 1\cdot\frac{1}{2} + 2\cdot\frac{1}{4} + 4\cdot\frac{1}{8} + \dots + 2^{k-1}\cdot\frac{1}{2^k}$

$$= \frac{k}{2} \xrightarrow[k\to\infty]{} \infty$$

(Also: $\log(2^k) = k\cdot \log 2 \xrightarrow[k\to\infty]{} \infty$
if we use this property...)

— Since $\log(\frac{1}{x}) = -\log x$

then $\lim_{x\to 0^+} \log x = -\infty$



$$\log : \mathbb{R}^+ \longrightarrow \mathbb{R}$$
$$x \longmapsto \log x$$

is bijective (injective (1-1) & surjective (onto))

— Injectivity:

$$\frac{d}{dx}\log x = \frac{1}{x} > 0 \quad \forall x \in \mathbb{R}^+$$

so log is strictly increasing:

$$x_1 > x_2 \implies \log x_1 > \log x_2$$

& also injective:

$$x_1 \neq x_2 \implies \log x_1 \neq \log x_2$$

$$\left(\text{or } \log x_1 = \log x_2 \implies x_1 = x_2 \right)$$

— Surjectivity:

log is continuous & $\log x \xrightarrow[x \to \infty]{} \infty$

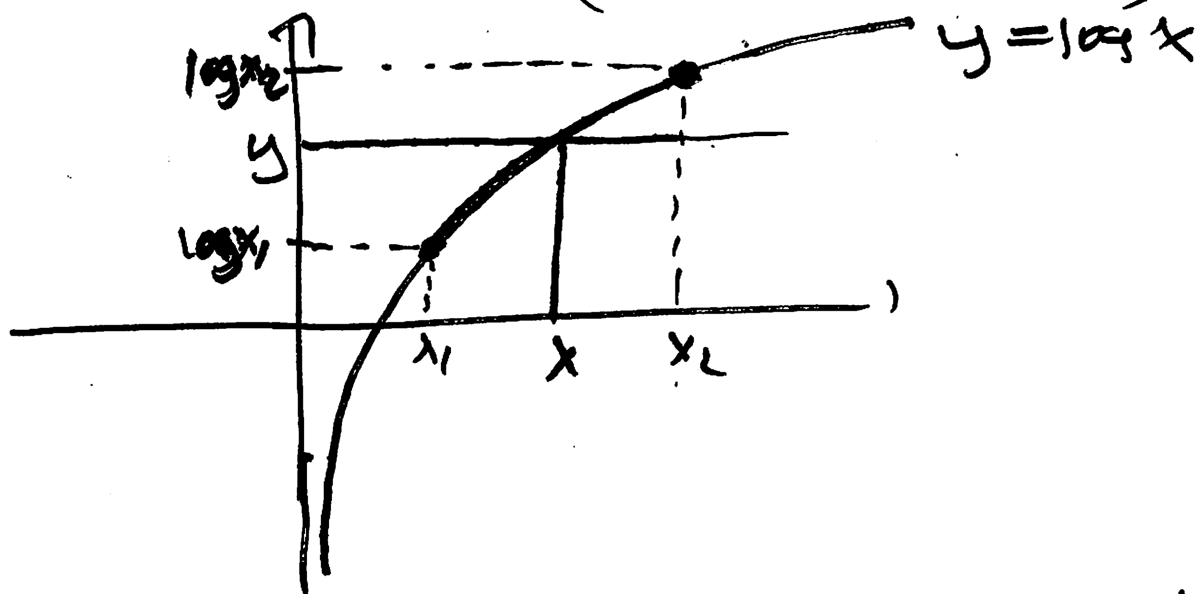
& $\log x \xrightarrow[x \to 0^+]{} -\infty$

— Let $y \in \mathbb{R}$ then by above we can find x_1 & x_2 such that

$$\log x_1 < y < \log x_2$$

Since log is continuous then $\log x$ takes any real value between $\log x_1$ & $\log x_2$

In particular $y \in [\log x_1, \log x_2]$ so there is an x with $\log x = y$ ($x \in [x_1, x_2]$)



Since log is bijective then it has an <u>inverse</u> function

$$\exp : \mathbb{R} \to \mathbb{R}^+$$
$$\exp(x) = \log^{-1}(x)$$

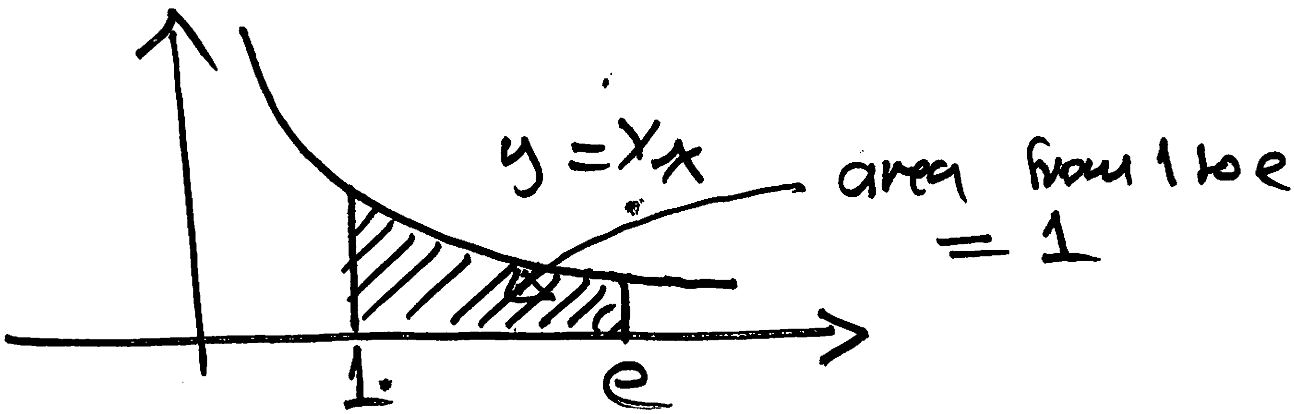
The <u>exponential function</u> $\exp(x) = e^x$ is defined by

$$\boxed{e^x := \log^{-1}(x)}$$

— In particular there is a unique real number $e \in \mathbb{R}$ given by

$$\boxed{e := \log^{-1}(1)}$$

$e \approx 2.718281828$



Since $f(f^{-1}(x)) = x$ & $f^{-1}(f(x)) = x$

then

- $e^{\log x} = x$ for all $x > 0$
- $\log(e^x) = x$ for all $x \in \mathbb{R}$

— If $y = e^x$ then

1. $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}} = \dfrac{1}{\frac{d}{dy}\log y} = \dfrac{1}{1/y} = y$

2. $y(0) = e^0 = \underline{1}$

— In fact, these are defining properties: $y = e^x$ is the only function with

1. $\frac{dy}{dx} = y$ & 2. $y(0) = 1$

ex Consider $x > -1$ $x \neq 0$

$$(1+x)^{1/x} = \left(e^{\log(1+x)}\right)^{1/x} = e^{\frac{\log(1+x)}{x}}$$

$$\lim_{x\to 0^+} \frac{\log(1+x)}{x} = \text{"}\frac{0}{0}\text{"}$$

$$= \lim_{x\to 0^+} \frac{\frac{d}{dx}\log(1+x)}{\frac{d}{dx}(x)} \overset{\text{L.Hospital}}{=}$$

$$\lim_{x\to 0^+} \frac{1/(1+x)}{1} = 1$$

So $$\lim_{x\to 0^+} (1+x)^{1/x} = \lim_{x\to 0^+} e^{\frac{\log(1+x)}{x}}$$

$$= e^1 = e$$

(since e^x is continuous @ $x=1$)

$$\boxed{(1+x)^{1/x} \xrightarrow[x\to 0^+]{} e}$$

letting $x = 1/T$

$$\boxed{\left(1+\frac{1}{T}\right)^T \xrightarrow[T\to\infty]{} e}$$

(ex) Letting T go through large integers N

$$\left(\frac{N+1}{N}\right)^N = \left(1+\frac{1}{N}\right)^N \xrightarrow[N\to\infty]{} e$$

$N = 1000$:

$$\left(\frac{1001}{1000}\right)^{1000} \approx 2.716924\ldots$$

- This can be used for <u>compounded interest</u>

- When an account with an initial amount A_0 has interest rate r (expressed as a decimal # so 15% would be $r = 0.15$) & its computed k times per year then after t years the amount would be

$$A(t) = A_0\Big[\underbrace{\left(1+\frac{r}{k}\right)\left(1+\frac{r}{k}\right)\cdots\left(1+\frac{r}{k}\right)}_{k \text{ times per year}}\Big]^t$$

$$= A_0 \left(1 + \frac{r}{k}\right)^{kt}$$

This is how the banks compute it. !