Taylor & Maclaurin Series

— Assume we have a function

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

which is valid & abs. conv. for $|x-c|<R$. How do the coefficients $a_0, a_1, a_2, \ldots$ relate to $f(x)$?

— Well, for starters:

$$f(c) = a_0 + a_1(c-c) + a_2(c-c)^2 + \cdots = a_0$$

so $\boxed{a_0 = f(c)}$

— Also, $f'(x) = \sum_{n=0}^{\infty} n a_n (x-c)^{n-1}$ & so

$$f'(c) = 1 \cdot a_1 + 2a_2(c-c) + 3a_3(c-c)^2 + \cdots$$

so $\boxed{a_1 = \frac{f'(c)}{1!}}$

— Further $f''(x) = \sum_{n=0}^{\infty} n(n-1) a_n (x-c)^{n-2}$

& so ...

$$f''(c) = \underline{2 \cdot 1 \cdot a_2 + 3 \cdot 2 \cdot a_3 (c-c) + 4 \cdot 3 \cdot a_4 (c-c)^2 + \dots}$$

so $\boxed{a_2 = \dfrac{f''(c)}{2!}}$

— In general

$$f^{(k)}(x) = \sum_{n=0}^{\infty} n(n-1)(n-2)\cdots(n-k+1)\, a_n (x-c)^{n-k}$$

$$= k!\, a_k + (k+1)k \cdots (2)\, a_{k+1} (x-c)$$
$$+ (k+2)(k+1)k \cdots (3)\, a_{k+2} (x-c)^2$$
$$+$$
$$\vdots$$

Hence $f^{(k)}(c) = k!\, a_k$ or

$$\boxed{a_k = \frac{f^{(k)}(c)}{k!}}$$

Therefore we obtain the Taylor Series of $f(x)$ about $x = c$:

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n}$$

— This Taylor series is valid for all x where the series is abs. conv. (i.e $|x-c| < R$)

— If $c=0$ then the series is called Maclaurin series:

$$\boxed{f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n}$$

or

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots$$

ex Find the Maclaurin series for $f(x) = e^x$

— Here $f'(x) = e^x$, $f''(x) = e^x, \ldots, f^{(n)}(x) = e^x$
& so $f^{(n)}(0) = e^0 = 1$ for all $n \geq 0$

$$\Rightarrow \quad e^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

that is . . .

$$\boxed{e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$

This series is valid for all $x \in \mathbb{R}$

since $\frac{|a_{n+1}|}{|a_n|} = \frac{\left|\frac{x^{n+1}}{(n+1)!}\right|}{\left|\frac{x^n}{n!}\right|} = \frac{|x|}{n+1} \xrightarrow[n\to\infty]{} 0$

for every fixed $x \in \mathbb{R}$.

ex Find the Maclaurin series for $\sin x$.

— Here
$$f(x) = \sin x$$
$$f'(x) = \cos x$$
$$f''(x) = -\sin x$$
$$f'''(x) = -\cos x$$
$$f^{(4)}(x) = \sin x$$

& the cycle begins again ....!

So we get ....

$$f(0) = \sin 0 = 0 \qquad f^{(4)}(0) = \sin 0 = 0$$
$$f'(0) = \cos 0 = 1 \qquad f^{(5)}(0) = \cos 0 = 1$$
$$f''(0) = -\sin 0 = 0 \qquad f^{(6)}(0) = -\sin 0 = 0$$
$$f'''(0) = -\cos 0 = -1 \qquad f^{(7)}(0) = -\cos 0 = -1$$

— So for $\sin x$ we have

$$f(x) = \sin x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

$$= \frac{f(0)}{0!} + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3$$
$$+ \frac{f^{(4)}(0)}{4!} x^4 + \frac{f^{(5)}(0)}{5!} x^5 + \frac{f^{(6)}(0)}{6!} x^6 + \frac{f^{(7)}(0)}{7!} x^7$$
$$+ \frac{f^{(8)}(0)}{8!} x^8 + \dots$$

$$= \frac{1}{1!} x + \frac{(-1)}{3!} x^3 + \frac{1}{5!} x^5 + \frac{(-1)}{7!} x^7 + - + - + - \dots$$

or

$$\boxed{\begin{aligned} \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \\ &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - + - \dots \end{aligned}}$$

& this equality is valid for all $x \in \mathbb{R}$ ...

since
$$\frac{|a_{n+1}|}{|a_n|} = \left| \frac{\frac{(-1)^{n+1}}{(2n+3)!} x^{2n+3}}{\frac{(-1)^n}{(2n+1)!} x^{2n+1}} \right| = \frac{|x|^2}{(2n+3)(2n+2)} \xrightarrow[n \to \infty]{} 0$$

for every fixed $x \in \mathbb{R}$.

ex From the above we obtain

$$\cos x = \frac{d}{dx}(\sin x)$$
$$= \frac{d}{dx}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - + \cdots\right)$$
$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - + - \cdots$$

& so
$$\boxed{\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - + - \cdots}$$

& this equality is also valid for all $x \in \mathbb{R}$!

So:

- Series for $\frac{1}{1-x}$, $\frac{1}{1+x}$, $\log(1+x)$, $\arctan x$ are only valid for $|x|<1$.
- Series for e^x, $\sin x$, $\cos x$ are valid for all $x \in \mathbb{R}$.

ex Find the Taylor series for $x \cdot \sin(x^2)$ @ $x=0$ (i.e Maclaurin series)

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - + - \cdots$$

$$\Rightarrow \sin(x^2) = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + - + - \cdots$$

$$\Rightarrow x\sin(x^2) = x\left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + - + - \cdots\right)$$

$$= x^3 - \frac{x^7}{3!} + \frac{x^{11}}{5!} - \frac{x^{15}}{7!} + - + - \cdots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (x^2)^{2n+1} \cdot x$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{4n+3}.$$

– The great thing about Taylor series is their applications:

ex Find $\lim_{x \to 0} \dfrac{x\sin(x^2) - x^3}{\arctan x - x + \frac{x^3}{3} - \frac{x^5}{5}}$

– Using Taylor series @ $x=0$ we get:

$$\frac{x\sin(x^2) - x^3}{\arctan x - x + x^3/3 - x^5/5}$$

$$= \frac{\left(x^3 - \frac{x^7}{3!} + \frac{x^{11}}{5!} - \frac{x^{15}}{7!} + - + - \cdots\right) - x^3}{\left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \frac{x^9}{9} - + - \cdots\right) - \left(x - \frac{x^3}{3} + \frac{x^5}{5}\right)}$$

$$= \frac{-\frac{x^7}{3!} + \frac{x^{11}}{5!} - \frac{x^{15}}{7} + - + - \cdots}{-\frac{x^7}{7} + \frac{x^9}{9} - + - + \cdots}$$

$$= \frac{x^7\left(-\frac{1}{3!} + \frac{x^4}{5!} - \frac{x^8}{7!} + - + - \cdots\right)}{x^7\left(-\frac{1}{7} + \frac{x^2}{4} - + - + \cdots\right)}$$

$$= \frac{-\frac{1}{3!} + \frac{x^4}{5!} - \frac{x^8}{7!} + - + - \cdots}{-\frac{1}{7} + \frac{x^2}{9} - + - + \cdots} \xrightarrow[x \to 0]{} \frac{-\frac{1}{3!} + 0}{-\frac{1}{7} + 0}$$

$$= \frac{7}{3!} = \frac{7}{6}.$$

This would have been tedious w/LH
(differentiating $x\sin(x^2)$ 7 times is no fun...)

— What does the Taylor series say about the function ?

$\boxed{\text{Fact}}$ For $f(x)$ that is $(n+1)$ differentiable at some interval I containing $c \in \mathbb{R}$ we have

$$f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

remainder such that ...

$$\ldots \quad R_n(x) = \frac{f^{(n+1)}(\alpha)}{(n+1)!}(x-c)^{n+1}$$

for some # $\alpha \in I$ (between x & c)..

– So in general if:

$$|R_n(x)| = \left| \frac{f^{(n+1)}(\alpha)}{(n+1)!}(x-c)^{n+1} \right|$$

$$= \left| \frac{f^{(n+1)}(\alpha)}{(n+1)!} \right| \cdot |x-c|^{n+1} \xrightarrow[n \to \infty]{} 0$$

then

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n \quad \text{for all } x \in \mathbb{R}$$

ex For $f(x) = e^x$ we have $f^{(n+1)}(x) = e^x$

& so $R_n(x) = \dfrac{f^{(n+1)}(\alpha)}{(n+1)!} x^n = \dfrac{e^{\alpha} \cdot x^n}{(n+1)!}$

$\alpha \in]0, x[$,

or $$\left|R_n(x)\right| = \frac{|e^{\alpha}|\cdot|x|^n}{(n+1)!} < \frac{e^{x}\cdot|x|^n}{(n+1)!}$$

$$= e^{x}\cdot\left(\frac{|x|^n}{(n+1)!}\right) \xrightarrow[n\to\infty]{} 0.$$

Since (A integer) $$\frac{A^n}{n!} = \frac{A\cdot A\cdot A\cdots A}{1\cdot 2\cdot 3\cdots A}\cdot\frac{A\cdot A\cdots\cdots A}{(A+1)(A+2)\cdots\ (n)}$$

$$= \frac{A^A}{A!}\cdot\frac{A^{n-A}}{(A+1)\cdots(n-1)\cdot n} \leq \frac{A^A}{A!}\cdot\frac{A^{n-A-1}}{A^{n-A-1}}\cdot\frac{A}{n}$$

$$= \frac{A^A}{A!}\cdot\frac{A}{n} \xrightarrow[n\to\infty]{} 0.$$

Hence
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

is valid for all $x\in\mathbb{R}$ (consistent w/ RaT)

— However, here we have an estimate on the actual error $R_n(x)$!

— If $f(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots$

then

$$P_n(x) = f(c) + \frac{f'(c)}{1!}(x-c) + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n$$

is the <u>n-th Taylor polynomial of $f(x)$ @ $x = c$</u>.

— This polynomial is the one that best approximates $f(x)$ around $x = c$ among <u>all</u> polynomials of degree n:

ex Find the 2nd degree polynomial that approximates e^x the best around $x = 1$.

— $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$, $f(x) = e^x$

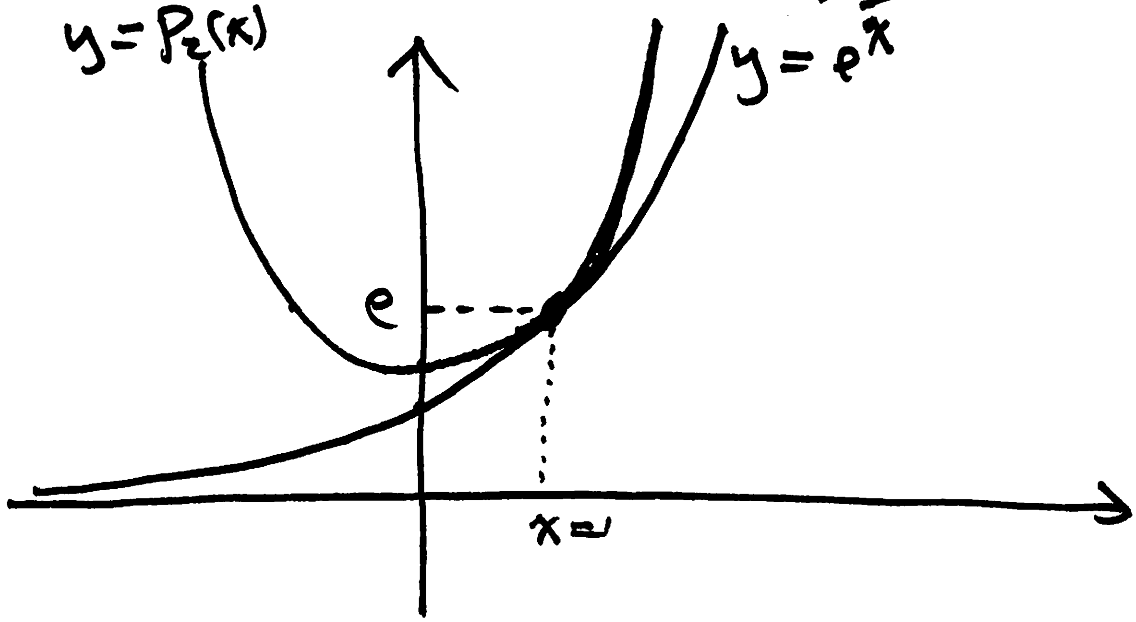
& so $f^{(n)}(x) = e^x$ for all $x \in \mathbb{R}$

$\Rightarrow f^{(n)}(1) = e$ for all n

$$\Rightarrow f(x) = \sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n$$

$$= e + \frac{e}{1!}(x-1) + \frac{e}{2!}(x-1)^2 + \cdots$$

So
$$P_2(x) = e + e(x-1) + \frac{e}{2}(x-1)^2$$
$$= \frac{e}{2}x^2 + \frac{e}{2} = \frac{e}{2}(x^2+1).$$



Further applications:

– Consider the polynomial $f(x) = (1+x)^m$.
In this case we have

$$f'(x) = m(1+x)^{m-1}$$
$$f''(x) = m(m-1)(1+x)^{m-2}$$
$$f'''(x) = m(m-1)(m-2)(1+x)^{m-3}$$
$$\vdots$$
$$f^{(k)}(x) = m(m-1)\cdots(m-k+1)(1+x)^{n-k}.$$

Using this we obtain

$$(1+x)^m = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n.$$

Since $$\frac{f^{(n)}(0)}{n!} = \frac{m(m-1)\cdots(m-n+1)}{n!}$$
$$= \frac{m!}{n!(m-n)!}$$

denoted by $\binom{m}{n}$ & read "m choose n".

$\boxed{\text{Fact}}$ Binomial Theorem

$$(1+x)^m = \sum_{n=0}^{\infty} \binom{m}{n} x^n \qquad \circledast$$

- If $m = 0,1,2,3,\dots$ then RHS of $\circledast$ is a polynomial, that is

$$(1+x)^m =$$

$$\binom{m}{0} + \binom{m}{1} x + \binom{m}{2} x^2 + \dots + \binom{m}{m-1} x^{m-1} + \binom{m}{m} x^m$$

where $\displaystyle \binom{m}{n} = \frac{m!}{n!(m-n)!}$

& this is valid for all $x \in \mathbb{R}$.

$\left(\text{since } \binom{m}{n} = 0 \text{ for } n > m \right)$

- Otherwise ($m \neq 0,1,2,\dots$) $\circledast$ holds for $|x| < 1$.

ex $\underline{m=5:}$

$$(1+x)^5 = \binom{5}{0} + \binom{5}{1}x + \binom{5}{2}x^2 + \binom{5}{3}x^3 + \binom{5}{4}x^4 + \binom{5}{5}x^5$$

$$= \frac{5!}{0!5!} + \frac{5!}{1!4!}x + \frac{5!}{2!3!}x^2 + \frac{5!}{3!2!}x^3 + \frac{5!}{4!1!}x^4 + \frac{5!}{5!0!}x^5$$

$$= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5.$$

Note the symmetry: $\binom{m}{n} = \binom{m}{m-n}$

— These coefficients can also be obtained by <u>Pascal's Triangle</u>:

$$\begin{array}{ccccccccccccl}
 & & & & & 1 & & & & & & \cdots & (1+x)^0 \\
 & & & & 1 & & 1 & & & & & \cdots & (1+x)^1 \\
 & & & 1 & & 2 & & 1 & & & & \cdots & (1+x)^2 \\
 & & 1 & & 3 & & 3 & & 1 & & & & \vdots \\
 & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1 & \cdots & (1+x)^5
\end{array}$$

$$\cdots\ a \qquad b\ \cdots$$
$$a+b$$

(ex) $m=-1$:

Note that $\binom{m}{0} = 1$ always !
$\binom{m}{1} = m$ always !
$\binom{m}{2} = \frac{m(m-1)}{2!}$ always !

So $\binom{-1}{0} = 1$, $\binom{-1}{1} = -1$ &

$$\binom{-1}{2} = \frac{(-1)(-1-1)}{2!} = \frac{1\cdot 2}{1\cdot 2} = 1$$

& in general

$$\binom{-1}{n} = \frac{(-1)(-1-1)\cdots(-1-n+1)}{n!}$$

$$= \frac{(-1)(-2)(-3)\cdots(-n)}{n!}$$

$$= \frac{(-1)^n\cdot n!}{n!} = \underline{\underline{(-1)^n.}}$$

Hence $$(1+x)^{-1} = \sum_{n=0}^{\infty}\binom{-1}{n}x^n$$

$$= \sum_{n=0}^{\infty}(-1)^n x^n$$

... which is consistent w/ geometric series

$$(1+x)^{-1} = \frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - + - + \dots$$

ex $m = \frac{1}{2}$:

$$(1+x)^{1/2} = \sqrt{1+x} = \sum_{n=0}^{\infty} \binom{1/2}{n} x^n.$$

$$\binom{1/2}{0} = 1$$

$$\binom{1/2}{1} = \frac{1}{2}$$

$$\vdots$$

$$\binom{1/2}{n} = \frac{(1/2)(1/2-1)(1/2-2)\cdots\cdots(1/2-n+1)}{n!}$$

$$= \frac{\frac{1}{2^n}(1)(-1)(-3)(-5)\cdots(-2n+3)}{n!}$$

$$= \frac{\frac{(-1)^{n-1}}{2^n}(1)(3)(5)\cdots(2n-3)}{n!}$$

$$= \frac{\frac{(-1)^{n-1}}{2^n}\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5\cdots\cdots(2n-3)(2n-2)}{2\cdot 4\cdot 6\cdots(2n-2)\cdot n!}$$

$$= \frac{(-1)^{n-1}(2n-2)!}{2^n\cdot 2^{n-1}(n-1)!\cdot n!}$$

$$= \frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}(n-1)!\,n!}$$

So

$$\sqrt{1+x} = (1+x)^{1/2} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}(n-1)!\,n!} x^n.$$

(ex) ((cont.)

- Find the 4^{th} order polynomial that best approximates $\sqrt{1+x}$ when $|x|$ is small (i.e. around $x=0$.)

- By Binomial Thm we have

$$\sqrt{1+x} = 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}(2n-2)!}{2^{2n-1}(n-1)!\,n!} x^n$$

So

$$P_4(x) = 1 + \frac{x}{2} + \frac{(-1)^1(4-2)!}{2^3 \cdot 1!\,2!} x^2 + \frac{(-1)^2(6-2)!}{2^5 \cdot 2!\,3!} x^3 + \frac{(-1)^3(8-2)!}{2^7 \cdot 3!\,4!} x^4$$

$$= 1 + \frac{x}{2} - \frac{2!}{8 \cdot 2} x^2 + \frac{4!}{2^6 \cdot 3!} x^3 - \frac{6!}{2^7 \cdot 3!\,4!} x^4$$

$$= 1 + \frac{x}{2} - \frac{x^2}{8} + \frac{x^3}{16} - \frac{5x^4}{128}.$$

(ex) Find the best 6^{th} degree polynomial approximation for $\sqrt{1-x^3}$ for small x.

— Since $\sqrt{1+x} \simeq 1 + \frac{x}{2} - \frac{x^2}{8}$

then $\sqrt{1-x^3} = \sqrt{1+(-x^3)}$

$$= 1 + \frac{(-x^3)}{2} - \frac{(-x^3)^2}{8}$$

$$= 1 - \frac{x^3}{2} - \frac{x^6}{8}.$$

— Since now $\sqrt{1-x^3} = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \cdots$

this tells us that:

$f(0) = 1$, $f'(0) = 0$, $f''(0) = 0$

$\frac{f'''(0)}{3!} = -\frac{1}{2}$, or $f'''(0) = -\frac{3!}{2} = 3.$

— This very idea can be pushed much further.

ex For $f(x) = x^3 \sin(x^4)$

find $f^{(999)}(0)$ & $f^{(1000)}(0)$

— since $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$

then

$$\sin(x^4) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^4)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+4}}{(2n+1)!}$$

& so $x^3 \sin(x^4) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+4} \cdot x^3}{(2n+1)!}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{8n+7}}{(2n+1)!}$$

— Looking @ the coefficient for x^{8n+7} –

on one hand $\frac{(-1)^n}{(2n+1)!}$,

on the other $\frac{f^{(8n+7)}(0)}{(8n+7)!}$,

we get . . . .

a) $8n+7=999$, $8n=992$, $\underline{\underline{n=124}}$:

$\Longrightarrow$ for $n=124$ we got:

$$\frac{f^{(999)}(0)}{999!}=\frac{(-1)^{124}}{(2\cdot 124+1)!}$$

or $$f^{(999)}(0)=\frac{(-1)^{124}}{249!}\cdot 999!=\underline{\underline{\frac{999!}{249!}}}.$$

b) $8n+7=1000$, $8n=993$

not div. by 8 !

$$n=\frac{993}{8}\neq \text{integer}$$

$$\Longrightarrow f^{(1000)}(0)=\underline{\underline{0.}} \; !$$