On Powers of some Intersection Graphs

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Abstract

We first consider *m-trapezoid graphs* and *circular m-trapezoid graphs* and give new constructive proofs that both these classes are closed under taking powers. We then consider general chordal graphs and present short and constructive proofs of the known fact that any odd power of a chordal graph is again chordal. We define a composition $(G, G') \mapsto G**G'$ of graphs which will yield an $O(n \log k)$ algorithm to obtain the representation of G^k if k is an odd positive integer and G is a chordal graph on n vertices with a given representation.

2000 MSC: 05C12, 05C62, 05C75, 05C85, 11B13, 26A15

Keywords: Power, intersection graph, m-trapezoid graph, circular m-trapezoid graph, chordal graph, composition of graphs.

1 Introduction

In this article we will study powers of some intersection graphs. Important classes of intersection graphs include interval graphs, circular arc graphs, trapezoid graphs and chordal graphs. In the first part we will focus our attention on m-trapezoid graphs, a class of intersection graphs, first defined in [1], that are a natural generalization of interval graphs and trapezoid graphs. We will show that any power of an m-trapezoid graph is again an m-trapezoid graph by explicitly constructing a representation for the power graph, from the representation of the original m-trapezoid graph. Interestingly enough, although this result is purely combinatorial, this new constructive proof makes use of some elementary real analysis. We conclude this part by stating the same results for circular m-trapezoid graphs.

In the second part we consider general chordal graphs and their powers. Chordal graphs and some subclasses of them have been studied extensively. For a good overview of recent work and references we refer to the introduction of [2]. In this second part we give some new, short and constructive

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proofs of the known fact that any odd power of a chordal graph is again chordal. Various proofs have been given, P. Duchet [3] showed that if G^k is chordal then so is G^{k+2} for any natural number k. A direct proof was also given by R. Balakrishnan and P. Paulraja in [4], which uses exhaustive case analysis. An elementary proof was also given in [2], which is based on edge contraction of simple graphs and the Euler formula for simple planar graphs. For our purposes here, we will rely on the result on F. Gavril [5], that the chordal graphs are precisely the intersection graphs of subtrees of a given tree. We define a *two-fold* composition $(G, G') \mapsto G**G'$ of graphs, which turns out to be neither associative nor commutative in general, but is associative when we consider all odd powers of a fixed chordal graph. This composition can then be used to yield an $O(n \log k)$ algorithm to compute the representation of G^k, if k is an odd positive integer and G is a chordal graph on n vertices with a given representation.

We will denote the positive integers $\{1, 2, 3, \ldots\}$ by $\mathbf{N}$, and the nonnegative integers $\{0, 1, 2, \ldots\}$ by $\mathbf{N}_0$. The set of real numbers (or *the real line*) will be denoted by $\mathbf{R}$. The closed interval $\{x : a \leq x \leq b\}$ will be denoted by $[a; b]$. The set of ordered tuples of real numbers, $\mathbf{R} \times \mathbf{R}$, (or *the real Euclidean plane*), will be denoted by $\mathbf{R}^2$. Similarly *the real d-dimensional Euclidean space* will be denoted by $\mathbf{R}^d$. All graphs we consider in this article are simple and undirected unless otherwise clearly stated. If G is a graph then we denote the set of vertices of G by $V(G)$, and the set of the edges by $E(G)$. For a given vertex $u \in V(G)$ the *open neighborhood* or just the *neighborhood* of u is the set of neighbors of u in G, not including u. It will be denoted by $N_G(u)$ or simply by $N(u)$ when there is no danger of ambiguity. Similarly the *closed neighborhood* of u is the set of neighbors of u in G, including the vertex u itself. This set will be denoted by $N_G[u]$ or simply by $N[u]$ when there is no ambiguity. Recall the following definition.

Definition 1.1 *Let G be a simple graph.*

1. *For vertices $u, v \in V(G)$ the* distance *between u and v in G, denoted by $d_G(u, v)$, is the number of edges in the shortest path in G that connects u and v.*

2. *For an integer $k \geq 1$, the* k-th power *of G is the graph G^k, where $V(G^k) = V(G)$ and*

$$E(G^k) = \{\{u, v\} : u, v \in V(G), u \neq v \text{ and } d_G(u, v) \leq k\}.$$

REMARK: We notice that G^0 is the graph with the same set of vertices as G and with no edges. G^1 is just G itself.

Recall the following definition of an intersection graph.

Definition 1.2 *A graph G has an* intersection representation $\{S_u : u \in V(G)\}$*, if it consists of a collection of sets that are in 1-1 correspondence with the vertices of G, in such a way that*

$$\{u, v\} \in E(G) \Leftrightarrow S_u \cap S_v \neq \emptyset.$$

In this case we call G an intersection graph *of* $\{S_u : u \in V(G)\}$.

Note that for an intersection graph G on n vertices, represented by sets $\{S_1, \ldots, S_n\}$, the *distance* $d(S_i, S_j)$ is just the distance between the corresponding vertices v_i and v_j in G.

As mentioned earlier in the introduction, both m-trapezoid graphs and chordal graphs are special kinds of intersection graphs, which we will consider in the next two sections, the rest of this article.

2 Trapezoid graphs

In this section we consider m-trapezoid graphs. Just like an interval graph is an intersection graph of a set of closed intervals of the real line $\mathbf{R}$, an m-trapezoid graph is an intersection graph of a set of m-trapezoids in the real plane $\mathbf{R}^2$. Assume that for each $l \in \{0, 1, \ldots, m\}$ we have two real numbers, a_l and b_l with $a_l < b_l$. As defined in [1], an *m-trapezoid* T is simply the closed interior of the polygon formed by the points

$$S = \{(a_l, l), (b_l, l) : l \in \{0, 1, \ldots, m\}\} \subseteq \mathbf{R}^2.$$

We denote that by $T = \mathrm{inter}(S)$. An *m-trapezoid graph* is a graph G which is an intersection graph of a set of m-trapezoids, that is, the vertices $V(G) = \{v_1, \ldots, v_n\}$ of G can be put in 1-1 correspondence with a set $\{T_1, \ldots, T_n\}$ of m-trapezoids in such a way that

$$\{v_i, v_j\} \in E(G) \Leftrightarrow T_i \cap T_j \neq \emptyset.$$

Let G be an m-trapezoid graph, represented by $\{T_1, \ldots, T_n\}$ where each

$$T_i = \mathrm{inter}(\{(a_{li}, l), (b_{li}, l) : l \in \{0, 1, \ldots, m\}\}) \qquad (1)$$

is the interior of the polygon formed by the points indicated. Whether the intersection $T_i \cap T_j$ is nonempty or not, only depends on conditions on the numbers a_{li}, b_{li}, a_{lj}, b_{lj} where $l \in \{0, 1, \ldots, m\}$. Hence, if $f : \{0, 1, \ldots, m\} \to \mathbf{R}$ is *any* injective function, then $\{T'_1, \ldots, T'_n\}$ where

$$T'_i = \mathrm{inter}(\{(a_{li}, f(l)), (b_{li}, f(l)) : l \in \{0, 1, \ldots, m\}\}), \qquad (2)$$

is also a representation of G. In some sense, the m-trapezoid representation of G depends only on the set of intervals $[a_{li}; b_{li}]$. In fact, as shown in [1], the class of m-trapezoid graphs can be categorized as precisely the class of co-comparability graphs of an order P with interval dimension $m+1$ or less. For simplicity we may as well assume $f(l) = l$ for each $l \in \{0, 1, \ldots, m\}$ in our representation, unless otherwise clearly stated. We also will write $\tilde{a}_{li}$ (resp. $\tilde{b}_{li}$) for the point (a_{li}, l) (resp. (b_{li}, l)) in $\mathbf{R}^2$.

For an integer $k > 1$, Carsten Flotow showed in [1] that if G^{k-1} is an m-trapezoid graph, then so is G^k, thereby proving that any power of an m-trapezoid graph is also an m-trapezoid graph. Our first goal is to construct an explicit m-trapezoid representation for G^k, when a representation for G as in (1) is given. Let each T_i be the closed region as in (1). For $k \geq 1$ let

$$b_{li}(k) = \max_{d(T_\alpha, T_i) \leq k-1} \{b_{l\alpha}\}$$

for every $i \in \{1, \ldots, n\}$ and $l \in \{0, \ldots, m\}$. Let likewise $\tilde{b}_{li}(k)$ be the point $(b_{li}(k), l)$ according to our previous convention. Define a new set of m-trapezoids as

$$T_i(k) = \mathrm{inter}(\{\tilde{a}_{li}, \tilde{b}_{li}(k) : l \in \{0, \ldots, m\}\}),$$

for each $i \in \{1, \ldots, n\}$. With this setup we have the following:

Theorem 2.1 *If G is an m-trapezoid graph represented by $\{T_1, \ldots, T_n\}$ then G^k is also an m-trapezoid graph, represented by $\{T_1(k), \ldots, T_n(k)\}$.*

Before we prove this theorem, we make note of the following useful observation, which explains the connection mentioned here above between the combinatorial object of a graph and some elementary real analysis.

Observation 2.2 *For an intersection graph G of topologically closed and bounded regions $\{C_1, \ldots, C_n\}$ of $\mathbf{R}^2$, the following statements are equivalent:*

1. *$d(C_i, C_j) \leq k$.*
2. *For every pair of points, $\tilde{x}_i \in C_i$ and $\tilde{x}_j \in C_j$, there is a continuous path joining $\tilde{x}_i$ and $\tilde{x}_j$, whose graph is entirely contained in the union of at most $k+1$ of the regions, two of them being C_i and C_j.*

We will also say that the left sides of T_i and T_j *cross* or *intersect* if there are two distinct numbers $p, q \in \{0, \ldots, m\}$ such that $a_{pi} < a_{pj}$ and $a_{qi} > a_{qj}$.

Proof. To prove Theorem 2.1 we need to show

$$d(T_i, T_j) \leq k \Leftrightarrow T_i(k) \cap T_j(k) \neq \emptyset. \tag{3}$$

In the case where the left sides of T_i and T_j, L_i and L_j respectively, cross we have that both $T_i \cap T_j$ and $T_i(k) \cap T_j(k)$ are nonempty, and there is nothing to prove. Hence, we can assume the left sides of T_i and T_j do not cross, say $a_{li} < a_{lj}$ for all $l \in \{0, \ldots, m\}$.

To prove the first implication of (3) assume $d(T_i, T_j) \leq k$. This means that there is an m-trapezoid T_α with $d(T_i, T_\alpha) \leq k - 1$ and $T_\alpha \cap T_j \neq \emptyset$, and hence there is an $l \in \{0, \ldots, m\}$ with $a_{lj} < b_{l\alpha}$. For this l we now have

$$a_{li} < a_{lj} < b_{l\alpha} \leq b_{li}(k),$$

by the mere definition of $b_{li}(k)$, and hence $T_i(k) \cap T_j(k) \neq \emptyset$.

To prove the second implication of (3) assume that $T_i(k) \cap T_j(k) \neq \emptyset$. Since by our assumption $a_{li} < a_{lj}$ for all $l \in \{0, \ldots, m\}$ there must be an $l \in \{0, \ldots, m\}$ such that $a_{lj} < b_{li}(k)$. Let T_β be an m-trapezoid with $d(T_i, T_\beta) \leq k - 1$ and $b_{li}(k) = b_{l\beta}$.

Restricting our attention to the subset $\mathbf{R} \times [0; m]$ of $\mathbf{R}^2$, which contains all our m-trapezoids, we see that L_j, the left side of T_j, formed by the line segments that connect $\tilde{a}_{l-1\,j}$ and $\tilde{a}_{lj}$ for all $l \in \{1, \ldots, m\}$, forms two regions of $\mathbf{R} \times [0; m]$, one to the left of L_j, and the other to the right of L_j. By Observation 2.2 there is a continuous path γ connecting $\tilde{a}_{li}$ and $\tilde{b}_{l\beta}$ whose graph lies entirely in the union of at most k m-trapezoids, two of them being T_i and T_β. Since in particular, the graph of γ lies in $\mathbf{R} \times [0; m]$, with one endpoint $\tilde{a}_{li}$ to the left of L_j and the other endpoint $\tilde{b}_{l\beta}$ to the right of L_j, then by a classical intermediate principle of real analysis, γ and L_j must intersect at some point $\tilde{p}$ on L_j.

Given two points $\tilde{x}_i \in T_i$ and $\tilde{x}_j \in T_j$ let λ_i be a continuous path from $\tilde{x}_i$ to $\tilde{a}_{li}$ within T_i, and λ_j be a continuous path from $\tilde{p}$ to $\tilde{x}_j$ within T_j. We can now form a new continuous path γ' from $\tilde{x}_i$ to $\tilde{x}_j$, by traversing from $\tilde{x}_i$ to $\tilde{a}_{li}$ along λ_i, from $\tilde{a}_{li}$ to $\tilde{p}$ along γ, and from $\tilde{p}$ to $\tilde{x}_j$ along λ_j. Now γ' is a continuous path connecting $\tilde{x}_i \in T_i$ and $\tilde{x}_j \in T_j$, whose graph lies within the union of T_j and at most k other m-trapezoids, one of those being T_i. Therefore the graph of γ' is contained in the union of at most $k + 1$ m-trapezoids, two of them being T_i and T_j. By Observation 2.2 we have $d(T_i, T_j) \leq k$, which proves the second implication of (3), and hence our theorem. □

Consider an m-trapezoid graph G represented by $\{T_1, \ldots, T_n\}$ where each T_i is given by (1). As we saw earlier $\{T'_1, \ldots, T'_n\}$ is also a representation of G where T'_i is given by (2) and $f(l) = l + 1$. We can also by simple horizontal translation and scaling assume that $0 < a_{li} < b_{li} < 2\pi$ for each $i \in \{1, \ldots, n\}$ and $l \in \{0, \ldots, m\}$. If we now consider the map $c : \mathbf{R}^2 \to \mathbf{R}^2$ defined by

$$c(x, y) = (y \cos(x), y \sin(x)),$$

then each m-trapezoid T, with upper and lower sides parallel to x-axis and the other sides some straight lines, is mapped to a *circular m-trapezoid*

$c(T)$, where the upper and lower sides are arcs of two co-centered circles, and the two remaining sides are segments of linear spirals. In this way we see that $\{c(T_1), \dots, c(T_n)\}$ is a *circular m-trapezoid representation* of G.

Definition 2.3 *A graph G is called a* circular m-trapezoid graph *if it is an intersection graph of a set $\{C_1, \dots, C_n\}$ of circular m-trapezoids in $\mathbf{R}^2$.*

Since the circular arc graphs are simply circular 0-trapezoid graphs we see in particular that the class of circular arc graphs and the class of m-trapezoid graphs both are contained in the class of circular m-trapezoid graphs.

If now G is an intersection graph of a set of circular m-trapezoids $\{C_1, \dots, C_n\}$ and $k \geq 1$ is an integer, then by analogously extending each C_i in the counterclockwise direction, as we extended each m-trapezoid T_i to the right to get $T_i(k)$, we likewise get a collection of circular m-trapezoids $\{C_1(k), \dots, C_n(k)\}$. We conclude this section with the following theorem, whose proof is nearly identical to that of Theorem 2.1.

Theorem 2.4 *If G is a circular m-trapezoid graph, represented by the set $\{C_1, \dots, C_n\}$, then G^k is also a circular m-trapezoid graph represented by the circular m-trapezoids $\{C_1(k), \dots, C_n(k)\}$.*

3 Powers of chordal graphs

In this section we consider powers of chordal graphs. Recall that a graph is chordal if and only if it is an intersection graph of a set of subtrees of a given tree [5]. For a chordal graph G only the odd powers of G are in general chordal. Let us start by using the characterization of F. Gavril to prove the following result of P. Duchet [3].

Theorem 3.1 *Let G be a graph and k a positive integer. If G^k is chordal then so is G^{k+2}.*

Proof. Assume that G^k is chordal on $V(G) = \{v_1, \dots, v_n\}$ and hence an intersection graph of a set of subtrees, $\{T_1, \dots, T_n\}$, of a given tree T, where each vertex v_i is represented by the subtree T_i. For each i let

$$T'_i = \bigcup_{v_\alpha \in N_G[v_i]} T_\alpha. \tag{4}$$

Each v_α in the above intersection is a neighbor of v_i in G and hence also in G^k. Therefore each $T_\alpha \cap T_i$ is nonempty. This means that each T'_i is a subtree of T rather than a disconnected sub-forest. We now show that G^{k+2} is an intersection graph of $\{T'_1, \dots, T'_n\}$.

Note that $T'_i \cap T'_j \neq \emptyset$ if and only if there are $v_\alpha \in N_G[v_i]$ and $v_\beta \in N_G[v_j]$ such that $T_\alpha \cap T_\beta \neq \emptyset$. Since now G^k is an intersection graph of $\{T_1, \dots, T_n\}$

this means precisely that $d_G(v_i, v_j) \leq k+2$, and hence that v_i and v_j are connected in G^{k+2}. Therefore G^{k+2} is an intersection graph of subtrees of T, and is therefore chordal. □

From the the above proof we see that if we know the representation of G as an intersection graph of subtrees of a given tree, then we know the representation of G^3 as an intersection graph. From this we can get the representation of G^5, and so on, eventually obtaining a representation of G^k as an intersection graph, where k is odd.

This can be done more directly by using a slight variation of the construction in (4). Note that in that formula, $v_\alpha \in N_G[v_i]$ and $d_G(v_\alpha, v_i) \leq 1$ is the same condition.

With this in mind, assume now that G is chordal on n vertices and represented by the subtrees $\{T_1, \ldots, T_n\}$ of a given tree T. If k is an odd integer, $k = 2m+1$, define $T_i(k)$ by the following formula

$$T_i(k) = \bigcup_{d(T_\alpha, T_i) \leq m} T_\alpha. \tag{5}$$

We now have the following proposition.

Proposition 3.2 *If $k = 2m+1$ is an odd integer, G is chordal and represented by the subtrees $\{T_1, \ldots, T_n\}$, then G^k is chordal and represented by the subtrees $\{T_1(k), \ldots, T_n(k)\}$ from (5).*

Proof. We need to show two facts. Firstly that each $T_i(k)$ is actually a subtree of T, and secondly that the $T_i(k)$ actually represent G^k as an intersection graph.

Clearly each $T_i(k)$ is either a sub-forest or a subtree of T. The fact that it is a connected subtree rather than a sub-forest is a special case of Observation 2.2, if we view trees as compact and closed regions in $\mathbf{R}^2$. That is, if $d(T_\alpha, T_i) \leq m$, then there is a sequence

$$T_\alpha = T'_0, T'_1, \ldots T'_m = T_i$$

such that each intersection $T'_l \cap T'_{l+1}$ is nonempty. This means in particular that there is a path from any vertex in T_α to any vertex in T_i within the union defining $T_i(k)$, and hence $T_i(k)$ is connected.

Assume now that $T_i(k) \cap T_j(k) \neq \emptyset$. This means that there are $\alpha, \beta \in \{1, \ldots, n\}$ such that $T_\alpha \cap T_\beta \neq \emptyset$ where $d(T_\alpha, T_i), d(T_\beta, T_j) \leq m$. Hence we have

$$d(T_i, T_j) \leq d(T_i, T_\alpha) + d(T_\alpha, T_\beta) + d(T_\beta, T_j) \leq m + 1 + m = k.$$

The other direction works the same way, and hence we have the proposition. □

In Proposition 3.2 we get the representation of G^k *directly* as the mathematical formula (5) indicates, similar to the one in Theorem 2.1 for m-trapezoid graphs, and in contrast to the *inductive* formula (4). In (5) though, the difficulty is computational, of determining which subtrees T_α are of distance m or less from T_i, for each given fixed $i \in \{1, \dots, n\}$.

So a natural question is: Is there a way to find a representation of G^k as an intersection graph if we know the representation of G and k, that will minimize these computational difficulties?

For the rest of this article we will extract some common themes of the proofs of Theorem 3.1 and Proposition 3.2 to define a certain composition of graphs. That can then be used to present an algorithm to compute the representation of G^k as an intersection graph of subtrees of a given tree, where G is chordal and k odd, with minimal computational complexities. First we need some new definitions and easily stated facts.

Definition 3.3 *Let G and G' be graphs on the same set of vertices, $V(G) = V(G') = V$. Define the* two-fold *composition $G**G'$ as the graph with $V(G**G') = V$ and edge set*

$$\begin{aligned} E(G**G') &= \{\{u,v\} : u \neq v \textit{ and } \exists w, w' \in V \ni \\ &\quad \{u,w\}, \{w',v\} \in E(G') \cup V \textit{ and} \\ &\quad \{w,w'\} \in E(G) \cup V\}. \end{aligned}$$

Remarks: (i) the condition $x \neq y$ is only to eliminate possible loops and to make the graph $G**G'$ simple. (ii) Note that $\{x,y\} \in E(G) \cup V$ (resp. $\{x,y\} \in E(G') \cup V$) means that either $x = y$, or $\{x,y\}$ is an edge in G (resp. $x = y$, or $\{x,y\}$ is an edge in G'). (iii) Note also that $\{u,v\} \in E(G**G')$ if and only if there are $w \in N_{G'}[u]$ and $w' \in N_{G'}[v]$ with $\{w,w'\} \in E(G) \cup V$. (iv) When restricting to the class of interval graphs or m-trapezoid graphs, a similar composition

$$(G, G') \mapsto G * G',$$

but closer related to a usual group-like product, can be defined. For more details see [6].

Directly from the above definition we get

Observation 3.4 *If s and t are nonnegative integers and G is a graph, then $G^s**G^t = G^{s+2t}$.*

The proof of the following theorem is analogous to the proof of Theorem 3.1. It is the mere statement of this more general fact that is of value for us, and which captures the common concept of Theorem 3.1 and Proposition 3.2.

Theorem 3.5 *Let G and G' be graphs on the same set of n vertices. If G is an intersection graph of a set $\{C_1, \dots, C_n\}$ of topologically closed and* *bounded regions in some Euclidean space $\mathbf{R}^d$, then the two-fold composition $G**G'$ is an intersection graph of $\{C_1^{**}, \ldots, C_n^{**}\} \subseteq \mathbf{R}^d$ where*

$$C_i^{**} = \bigcup_{v_\alpha \in N_{G'}[v_i]} C_\alpha \tag{6}$$

for each $i \in \{1, \ldots, n\}$.

Proof. For distinct indices i and j we have that $C_i^{**} \cap C_j^{**} \neq \emptyset$ if and only if there are $v_\alpha \in N_{G'}[v_i]$ and $v_\beta \in N_{G'}[v_j]$ with $C_\alpha \cap C_\beta \neq \emptyset$. This means precisely that $\{v_\alpha, v_\beta\} \in E(G) \cup V(G)$ and the theorem follows. □

Needless to say, what is of interest to us, are those cases where the regions C_i^{**} belong to the same class as the C_i do. An example of that case is the following corollary.

Corollary 3.6 *Let G' be a subgraph of the chordal graph G on the same set of n vertices. If G is represented by the subtrees $\{T_1, \ldots, T_n\}$ of a tree T, then $G**G'$ is chordal and is represented by the subtrees $\{T_1^{**}, \ldots, T_n^{**}\}$ of T, where each T_i^{**} is given by (6).*

Proof. Since G' is a subgraph of G we have that every edge in G' is also in G, and hence we have that $T_\alpha \cap T_i \neq \emptyset$ whenever $v_\alpha \in N_{G'}[v_i]$. Hence T_i^{**} is connected and therefore a subtree of T. □

For a given graph G we have that G^t is always a subgraph of G^s if $t \leq s$. Hence, if G is a graph on n vertices such that both G^s and G^t are chordal and represented by subtrees $\{T_1(s), \ldots, T_n(s)\}$ and $\{T_1(t), \ldots, T_n(t)\}$ respectively, then $G^s**G^t = G^{s+2t}$ is chordal and represented by subtrees $\{T_1(s+2t), \ldots, T_n(s+2t)\}$, where for each i we have

$$T_i(s+2t) = \bigcup_{T_\alpha(t) \cap T_i(t) \neq \emptyset} T_\alpha(s). \tag{7}$$

From this we see that if we have the representations of G^s and G^t then we can *directly* get a representation of G^{s+2t}. Now the question really is: Given an odd integer k, how can we use (7) to obtain an algorithm yielding a representation of G^k which will be faster than an algorithm based on the proof of either Theorem 3.1 or Proposition 3.2?

In order to address this in a more precise manner, we need the following definition.

Definition 3.7 *A sequence $(\mathcal{T}_i)_{i \geq 0}$ of sets of positive integers with*

$$\begin{aligned} \mathcal{T}_0 &= \{1\}, \\ \mathcal{T}_{i+1} &= \mathcal{T}_i \cup \{s_i + 2t_i\}, \end{aligned}$$

for some $s_i, t_i \in \mathcal{T}_i$ *with* $t_i \leq s_i$*, is called a* two-fold set sequence*, or a* TFSS *for short.*

REMARKS: (i) Every TFSS starts with the set that contains just the integer 1, $\mathcal{T}_0 = \{1\}$. Every TFSS has the second set $\mathcal{T}_1 = \{1, 3\}$, since $s_0 = t_0 = 1$ is the only possibility. For the third set $\mathcal{T}_2$, there are three possibilities, $\mathcal{T}_2 = \{1, 3\}$ when $s_1 = t_1 = 1$, $\mathcal{T}_2 = \{1, 3, 5\}$ when $s_1 = 3$ and $t_1 = 1$, and $\mathcal{T}_2 = \{1, 3, 9\}$ when $s_1 = t_1 = 3$. (ii) We note that by definition, every set in a TFSS contains only odd integers. Also note that for every odd integer $k = 2m + 1$ there is a TFSS $(\mathcal{T}_i)_{i\geq 1}$ with $k \in \mathcal{T}_m$, namely $\mathcal{T}_i = \{1, 3, \ldots, 2i + 1\}$ where $s_i = 2i + 1$ and $t_i = 1$. The next lemma shows that one can reach any odd integer k in $O(\log k)$ steps. Recall that $\lfloor x \rfloor$ is the largest integer $n \leq x$, for any real x, and that the base-α logarithm is denoted by $\log_\alpha$.

Lemma 3.8 *For every odd positive integer* k *there is a TFSS* $(\mathcal{T}_i)_{i\geq 1}$ *with*

$$k \in \mathcal{T}_{4\lfloor \log_3(k) \rfloor}.$$

In order to prove the above lemma we need some tools.

Claim 3.9 *If* $N \geq 3^l$ *and* $\{1, 3, 3^2, \ldots, 3^l, N\} \subseteq \mathcal{T}_\gamma$ *then it is possible to have*

$$N + 3^p + 3^q \in \mathcal{T}_{\gamma+p-q+1}$$

for any $p, q \in \{0, 1, \ldots, l\}$ *with* $p \geq q$*.*

Proof. If $p = q$ we let $s_\gamma = N$, $t_\gamma = 3^q$ and get $N+3^p+3^q = N+2\cdot 3^q \in \mathcal{T}_{l+1}$ proving our claim in this case. If $p > q$ we note that

$$N + 3^p + 3^q = N + 2(3^{p-1} + 3^{p-2} + \cdots + 3^q) + 2 \cdot 3^q. \tag{8}$$

Define s_γ and t_γ as in the case $p = q$. For each $i \in \{1, \ldots, p - q - 1\}$, if nonempty, we let

$$\begin{aligned} s_{\gamma+i} &= N + 2(3^{q+i-1} + \cdots + 3^q), \\ t_{\gamma+i} &= 3^{q+i}. \end{aligned}$$

By letting $s_{\gamma+p-q} = N + 2(3^{p-1} + 3^{p-2} + \cdots + 3^q)$ and $t_{\gamma+p-q} = 3^q$, we get by (8) that $N + 3^p + 3^q \in \mathcal{T}_{\gamma+p-q+1}$ which concludes the proof of our claim. □

From the above claim we get two following corollaries.

Corollary 3.10 *If* $N \geq 3^l$ *and* $\{1, 3, 3^2, \ldots, 3^l, N\} \subseteq \mathcal{T}_\gamma$ *then for any subset* $Y \subseteq \{0, \ldots, l\}$ *it is possible to have*

$$N + 2\sum_{y\in Y} 3^y \in \mathcal{T}_{\gamma+|Y|}.$$

Proof. For each element $y \in Y$, let $p = q = y$, and apply Claim 3.9 repeatedly. Hence we get the corollary. □

Corollary 3.11 *If* $N \geq 3^l$ *and* $\{1, 3, 3^2, \ldots, 3^l, N\} \subseteq \mathcal{T}_\gamma$ *then for any subset* $X \subseteq \{0, \ldots, l\}$ *with* $|X|$ *even, it is possible to have*

$$N + \sum_{x \in X} 3^x \in \mathcal{T}_{\gamma+l+1}.$$

Proof. Let $|X|/2 = t$. Listing the elements of $X = \{x_1, x'_1, \ldots, x_t, x'_t\}$ in a strictly decreasing order, we can apply Claim 3.9 t times and get

$$\begin{aligned} N + \sum_{x \in X} 3^x &= N + (3^{x_1} + 3^{x'_1}) + \cdots + (3^{x_t} + 3^{x'_t}) \\ &\in \mathcal{T}_{\gamma+(x_1-x'_1+1)+\cdots+(x_t-x'_t+1)}. \end{aligned}$$

Since $X = \{x_1, x'_1, \ldots, x_t, x'_t\} \subseteq \{0, \ldots, l\}$, and the listing is strictly decreasing, we have that $(x_1 - x'_1 + 1) + \cdots + (x_t - x'_t + 1) \leq l + 1$. Therefore we have $N + \sum_{x \in X} 3^x \in \mathcal{T}_{\gamma+l+1}$, proving the corollary. □

At this point we can prove Lemma 3.8:

Proof. Consider the odd integer $k > 1$ written in base 3,

$$k = \sum_{a \in A} 3^a = 2 \sum_{b \in B} 3^b,$$

where $A, B \subseteq \mathbf{N}_0$, $A \cap B = \emptyset$ and $|A|$ is odd. If $l = \lfloor \log_3(k) \rfloor$ then $l = \max(A \cup B)$. Note that $l + 1 \geq |A \cup B| = |A| + |B|$.

We form the first $l+1$ terms of our TFSS by letting $s_i = t_i = 3^i$ for each $i = 0, 1, \ldots, l$ and thereby getting $\mathcal{T}_0, \ldots, \mathcal{T}_l$ where $\mathcal{T}_i = \{1, 3, 3^2, \ldots, 3^i\}$ for each i. At this point we proceed as indicated in the following two cases.

First case: The leading coefficient (i.e. the coefficient of 3^l) is 1, and hence $l \in A$. We have that $\mathcal{T}_l = \{1, 3, 3^2, \ldots, 3^l\}$, and letting $N = 3^l$ in Corollary 3.10 we get that we can have

$$\begin{aligned} k - \sum_{a \in A \setminus \{l\}} 3^a &= 3^l + 2 \sum_{b \in B} 3^b \\ &\in \mathcal{T}_{l+|B|}. \end{aligned}$$

Since $|A \setminus \{l\}|$ is even, we get from Corollary 3.11 that we can have

$$k \in \mathcal{T}_{(l+|B|)+l+1}.$$

Since $A \cup B \subseteq \{0, \ldots, l\}$ are disjoint, and $A \neq \emptyset$, we have that $|B| \leq l$ and hence $k \in \mathcal{T}_{3l+1} \subseteq \mathcal{T}_{4l}$, proving the lemma in this first case.

Second case: The leading coefficient of the base-3 expansion of k is a 2, and hence $l \in B$. We have $\mathcal{T}_l = \{1, 3, 3^2, \ldots, 3^l\}$. If $a' = \max A$, then by letting $N = 3^l$, Corollary 3.10 implies that we can have

$$2 \cdot 3^l - 3^{a'} = 3^l + 2(3^{l-1} + \ldots + 3^{a'}) \in \mathcal{T}_{2l-a'}.$$

Again, by Corollary 3.10, we can have

$$\begin{aligned} 2\sum_{b \in B} 3^b + 3^{a'} &= (2 \cdot 3^l - 3^{a'}) + 2 \sum_{b \in (B \setminus \{l\}) \cup \{a'\}} 3^b \\ &\in \mathcal{T}_{(2l-a')+|B|}. \end{aligned}$$

Since $|A \setminus \{a'\}|$ is even we get by Corollary 3.11 that we can have

$$k = (2\sum_{b \in B} 3^b + 3^{a'}) + \sum_{a \in A \setminus \{a'\}} 3^a \in \mathcal{T}_{(2l-a'+|B|)+l+1}.$$

Since $A \neq \emptyset$ and $|B| \leq l$, we have $k \in \mathcal{T}_{4l+1-a'} \subseteq \mathcal{T}_{4l}$, proving the lemma in this case, and thereby completing the proof of the lemma. □

Remark: It is clear that if k is an odd integer then there is no TFSS $(\mathcal{T}_i)_{i \geq 1}$ with $k \in \mathcal{T}_i$, where $i < \lfloor \log_3(k) \rfloor$, since the largest number that can possible be in each $\mathcal{T}_i$ is 3^i, which we can have only if we set $s_i = t_i = 3^i$ for all i. Hence, for any k, the smallest i such that $k \in \mathcal{T}_i$ satisfies $\lfloor \log_3(k) \rfloor \leq i \leq 4 \lfloor \log_3(k) \rfloor$, and therefore $i = \Theta(\log k)$.

Consider now our original question on representing a chordal graph as an intersection graph of subtrees of a tree. If we have a given chordal graph G on n vertices, and k is an odd integer, then by Lemma 3.8 we need to perform *at most* $4 \lfloor \log_3(k) \rfloor$ two-fold compositions

$$(G^s, G^t) \mapsto G^s ** G^t = G^{s+2t},$$

to obtain G^k, when we start with $G = G^1$. Hence, we need to form at most $4n \lfloor \log_3(k) \rfloor$ unions, as in (7), to acquire the representation of G^k as an intersection of subtrees of a tree. Hence we have the following corollary.

Corollary 3.12 *Given a chordal graph G on n vertices represented by subtrees $\{T_1, \ldots, T_n\}$ of a tree T as an intersection graph. If k is an odd integer then the representation $\{T_1(k), \ldots, T_n(k)\}$ of G^k as an intersection graph of subtrees of T, can be reached in $O(n \log k)$ steps.*

Acknowledgments

The author is grateful to Magnús M. Halldórsson for suggesting this direction of research.

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May 22, 2007