On multipartite posets

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Abstract

Let $m \geq 2$ be an integer. We say that a poset $\mathbf{P} = (X, \preceq)$ is *m-partite* if X has a partition $X = X_1 \cup \cdots \cup X_m$ such that (1) each X_i forms an antichain in $\mathbf{P}$, and (2) $x \prec y$ implies $x \in X_i$ and $y \in X_j$ where $i, j \in \{1, \ldots, m\}$ and $i < j$. If $\mathbf{P}$ is m-partite for some $m \geq 2$, then we say it is *multipartite*. – In this article we discuss the order dimension of multipartite posets in general and derive tight asymptotic upper bounds on the order dimension of them in terms of their bipartite sub-posets.

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1 Introduction

This article was partly inspired by a question asked by Reinhard Laubenbacher [11] which casually can be phrased as follows: "For a given collection of posets, form a new poset by stacking them together, putting one on top of the other. Is it possible to bound the order dimension of the newly formed poset in terms of the order dimension of the given posets?" A precise definition of order dimension is given later in Section 2. Laubenbacher's motivation were posets that appeared in the following manner: When finitely many agents $A_1, \ldots, A_n$ are investigated over discrete times $t = 0, 1, \ldots, m$, one obtains a poset consisting of the $n(m+1)$ elements $A_i(t)$, where a directed edge from $A_i(t)$ down to $A_j(t+1)$ is present if, and only if, agent A_i has influenced agent A_j during the time interval from t to $t+1$. This resulting induced poset is sometimes called the *influence poset* among the agents. Note that here the maximal chains can have any length $\ell \in \{1, \ldots, m\}$ and that minimal and maximal elements can be at any time level t.

Other more familiar posets can also be viewed as stacked sub-posets, one on top of the other: If $\mathcal{F}_P$ is the face lattice of an n-dimensional polytope P and $\mathcal{F}_P(i, i+1)$ is the height-2 sub-poset of $\mathcal{F}_P$ consisting of the i and $(i+1)$-dimensional faces of P, then $\mathcal{F}_P$ can be thought of being formed by stacking $\mathcal{F}_P(1, 2)$ on top of $\mathcal{F}_P(0, 1)$, $\mathcal{F}_P(2, 3)$ on top of $\mathcal{F}_P(1, 2)$ and so on, finally stacking $\mathcal{F}_P(n-1, n)$ on top of $\mathcal{F}_P(n-2, n-1)$. In this case the stacking appears naturally since $\mathcal{F}_P$ is a *graded poset* provided with a grading function into the nonnegative integers, that maps each face of P (i.e. each element of the poset $\mathcal{F}_P$) to its dimension. (For more on graded posets see [15] and [13].) Determining the order dimension of face lattices of convex polytopes is hard. Some partial yet interesting results in this direction appear in [12] and later in [1]. Of particular interest is the face lattice of the standard n-simplex when viewed as the subset lattice of $\{1, \ldots, n\}$. If we

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let $[n] = \{1, \ldots, n\}$ and $\binom{[n]}{k}$ denotes all the k-element subsets of $[n]$, then the power set $\mathbb{P}([n])$ of all subsets of $[n]$ can be partitioned into $n+1$ disjoint sets $\mathbb{P}([n]) = \binom{[n]}{0} \cup \binom{[n]}{1} \cup \cdots \cup \binom{[n]}{n}$. For $0 \leq k_1 < k_2 \leq n$ denote the poset on $\binom{[n]}{k_1} \cup \binom{[n]}{k_2}$ induced by inclusion by $\mathbf{P}(k_1, k_2; n)$. Hence, as a poset, $\mathbb{P}([n])$ can be thought of being formed by stacking the n posets $\mathbf{P}(i, i+1; n)$ for $i \in \{0, 1, \ldots, n-1\}$ one on top of the other. Investigating the order dimension $\dim(k_1, k_2; n)$ of such sub-posets $\mathbf{P}(k_1, k_2; n)$ of $\mathbb{P}([n])$ for $1 \leq k_1 < k_2 \leq n-1$ is currently an active area of research, in particular the investigation of $\dim(1, k; n)$, the order dimension of the poset $\mathbf{P}(1, k; n)$. We list and briefly discuss a few related and celebrated results of this ongoing investigation:

In [2] an explicit formula for $\dim(1, k; n)$ is given when $2\sqrt{n} - 2 \leq k < n-1$ and in [17] the exact values of $\dim(1, 2; n)$ are given for $2 \leq n \leq 13$. In [9] and [5] the exact values for $\dim(2, n-2; n)$ and $\dim(k, n-k; n)$ are given, provided that certain conditions hold for k and n. In [14] the asymptotic behavior of $\dim(1, k; n)$ is given as a function of n when k is considered fixed. Finally, in [8] a direct method to determine $\dim(1, 2; n)$ for each n is given. Hence, the case $k = 2$ for determining $\dim(1, k; n)$ is the only case which can be considered completely solved. In [6] an $O(\Delta(\log \Delta)^2)$ upper bound is given for the order dimension of a poset, in which the number of elements comparable to any fixed element is at most Δ. The proof of this uses probabilistic methods, something that has turned out to be of great value for many asymptotic problems regarding order dimensions. In [10] however, it is shown by contradiction that $\dim(1, \log n, n) = \Omega(\log^3 n / \log(\log n))$. In addition, all the upper bounds derived there are proved by explicit construction, which therefore is also an effective method in providing bounds for order dimensions.

In what follows we will discuss a class of posets that will include the class of graded posets and the posets obtained by such "stacking" as mentioned above in an ad hoc manner. Our methods will be elementary and constructive. In addition, they will yield simpler proofs of well-known results.

2 Definitions and basic properties

By a *poset* $\mathbf{P}$ we will always mean an ordered tuple $\mathbf{P} = (X, \preceq)$ where $\preceq$ is a reflexive, antisymmetric and transitive binary relation on X. Unless otherwise stated X is always assumed to be a finite set. We will for the most part try to be consistent with the standard notation from [18]. In particular, if two elements $x, y \in X$ are incomparable in $\mathbf{P}$, then we write $x \parallel y$. By $\min(\mathbf{P})$ and $\max(\mathbf{P})$ we mean the set of minimal and maximal elements of $\mathbf{P}$ respectively. As originally defined in [3] and as stated in [18], the *order dimension* of $\mathbf{P} = (X, \preceq)$, denoted by $\dim(\mathbf{P})$, is the least number $d \in \mathbb{N}$ of linear extensions $\preceq_1, \ldots, \preceq_d$ of $\preceq$ that *realize* $\preceq$. This means that for $x, y \in X$ we have $x \preceq y$ in $\mathbf{P}$ iff $x \preceq_i y$ for all $i \in [d]$.

For $n \in \mathbb{N}$, any collection S of points in the n-dimensional Euclidean space $\mathbb{R}^n$ naturally forms a poset $(S, \preceq_E)$ by

$$\tilde{x} \preceq_E \tilde{y} \Leftrightarrow x_i \leq y_i \text{ for each } i \in [n],$$

for any $\tilde{x} = (x_1, \ldots, x_n)$ and $\tilde{y} = (y_1, \ldots, y_n)$ from S. With this in mind we have the following.

Observation 2.1 *Let $\mathbf{P} = (X, \preceq)$ be a poset. Its order dimension $\dim(\mathbf{P})$ is the least $d \in \mathbb{N}$ such that there is an injective homomorphism $\phi : \mathbf{P} \to \mathbb{R}^d$ satisfying $x \preceq y \Leftrightarrow \phi(x) \preceq_E \phi(y)$ for all $x, y \in X$.*

By a suitable translation, we may assume the the homomorphism ϕ from Observation 2.1 maps into $\mathbb{R}^d_+$, where every coordinate is positive. Hence, we see that $d = \dim(\mathbf{P})$ is the least positive integer such that $\mathbf{P}$ can be represented as a collection of d-dimensional boxes in $\mathbb{R}^d$ where each $x \in X$ is represented by the box spanned by the origin and $\phi(x)$ (each side of the box is parallel to one of the d axes of $\mathbb{R}^d$), and where the partial order is given by the corresponding containment order. (For more on geometric containment orders in general see [4].)

Determining the exact value of the order dimension of a poset is a hard computational problem. Even when we restrict to height-2 posets, the problem of computing their order dimensions is NP-complete [19].

Assume that $\mathbf{P}$ has n elements, $X = \{x_1, \dots, x_n\}$, and denote the standard basis for $\mathbb{R}^n$ by $\{\tilde{e}_1, \dots, \tilde{e}_n\}$. Consider $\phi : \mathbf{P} \to \mathbb{R}^n$ given by

$$\phi(x_i) = \sum_{x_\ell \preceq x_i} \tilde{e}_\ell. \tag{1}$$

If for any i, we let $D[x_i] = \{x_\ell \in X : x_\ell \preceq x_i\}$ be the *closed downset* of x_i in $\mathbf{P}$, then for any $i, j \in [n]$ we clearly have $x_i \preceq x_j \Leftrightarrow D[x_i] \subseteq D[x_j]$ and hence $x_i \preceq x_j \Leftrightarrow \phi(x_i) \preceq_E \phi(x_j)$, showing that ϕ is an injective homomorphism. We summarize in the following.

Observation 2.2 *For a poset* $\mathbf{P} = (X, \preceq)$ *with* $X = \{x_1, \dots, x_n\}$, *the map* ϕ *from (1) is an injective homomorphism into* $\mathbb{R}^n$. *In particular, we always have* $\dim(\mathbf{P}) \leq |X|$.

Remark: A theorem by Hiraguchi [7],[18],[13] states that the above Observation 2.2 can be improved by a factor of 1/2 so that $\dim(\mathbf{P}) \leq |X|/2$ for all posets $\mathbf{P}$ with $|X| \geq 4$. Moreover, for each $n \in \mathbb{N}$ the "standard example" $\mathbf{S}_{2n}$ from [3] and [18, p. 12], of a height-2 poset on $2n$ elements with order dimension of n, shows that the upper bound of Hiraguchi is tight as a function of $|X|$. The standard example $\mathbf{S}_{2n}$ is in fact the poset $\mathbf{P}(1, n-1; n)$ mentioned earlier in Section 1. However, the proof of Hiraguchi's tight upper bound is non-constructive and not completely trivial. The good thing about Observation 2.2 is that it is constructive and yields a direct $n \times n$ zero-one matrix representation of a poset $\mathbf{P} = (X, \preceq)$ with $X = \{x_1, \dots, x_n\}$, where each row i is given by $\phi(x_i)$ from (1).

3 Multipartite posets

In this section we address what can be said in general about the order dimension of a given poset in terms of its levels or induced sub-bipartite posets. We start with the following example that answers the question of Laubenbacher from Section 1 in the negative.

Example: For $n \in \mathbb{N}$ let $A = \{a_1, \dots, a_{2n}\}$, $B = \{b_1, \dots, b_{2n}\}$ and $C = \{c_1, \dots, c_{2n}\}$ be pairwise disjoint sets. Let $\mathbf{P}_a$ be the height-2 poset on $A \cup B$ where $a_i \succ b_{i+\ell}$ for each $i \in [2n]$ and $\ell \in \{0, \dots, n-1\}$, computed cyclically modulo $2n$. Likewise let $\mathbf{P}_b$ be the same height-2 poset on $B \cup C$ where $b_i \succ c_{i+\ell}$ for each $i \in [2n]$ and $\ell \in \{0, \dots, n-1\}$ modulo $2n$. Form the poset $\mathbf{P}_{a,b}$ on $A \cup B \cup C$ where the partial order is induced by those of $\mathbf{P}_a$ and $\mathbf{P}_b$. Note that both $\mathbf{P}_a$ and $\mathbf{P}_b$ are order isomorphic to the "generalized crown" $\mathbf{S}^{n-1}_{n+1}$ introduced by Trotter in [16]. There and in [18, p. 36] a closed formula for the order dimension $\dim(\mathbf{S}^k_n)$ is given. In particular we have

$$\dim(\mathbf{P}_a) = \dim(\mathbf{P}_b) = \dim(\mathbf{S}^{n-1}_{n+1}) = \left\lceil \frac{4n}{n+1} \right\rceil = 4.$$

However, we note that the poset $\mathbf{P}_{a,b}$ contains the standard example $\mathbf{S}_{2n}$ as an induced sub-poset, when restricted to the set $A \cup C$, and hence $\dim(\mathbf{P}_{a,b}) \geq 2n$. We summarize in the following observation.

Observation 3.1 *There is no function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that*

$$\dim(\mathbf{P}) \leq f(\dim(\mathbf{P}_1), \dim(\mathbf{P}_2))$$

holds in general for all posets $\mathbf{P}$, which are induced by two sub-posets $\mathbf{P}_1$ and $\mathbf{P}_2$ with $\min(\mathbf{P}_1) = \max(\mathbf{P}_2)$.

The following elementary lemma is a direct consequence of the *interpolation property* for posets [18, p. 21] and the fact that each poset has a linear extension.

Lemma 3.2 *Let $\mathbf{P}$ be a poset and $\mathbf{P}'$ be an induced sub-poset of $\mathbf{P}$. Then any linear extension $\mathbf{L}'$ of $\mathbf{P}'$ can be extended to a linear extension $\mathbf{L}$ of $\mathbf{P}$.*

Recall that a *bipartite* poset [18, p. 46] is an ordered triple $\mathbf{P} = (X, Y; \preceq)$ where X and Y are disjoint and $x \prec y$ implies that $x \in X$ and $y \in Y$. Note that any bipartite poset yields a unique height-2 poset, but not vice versa since isolated elements can be regarded as minimal or maximal but not both in a bipartite poset.

Definition 3.3 *Let $m \geq 2$ be an integer and $X_1, \ldots, X_m$ be disjoint nonempty sets. We call $\mathbf{P} = (X_1, \ldots, X_m; \preceq)$ an m-*partite poset *if $\preceq$ is a partial order on $X = X_1 \cup \cdots \cup X_m$ such that (1) each X_i forms an antichain w.r.t. $\preceq$, and (2) $x \prec y$ implies $x \in X_i$ and $y \in X_j$ where $i, j \in [m]$ and $i < j$. If $\mathbf{P}$ is m-partite for some m, then $\mathbf{P}$ is a* multipartite poset*.*

Clearly, each m-partite poset $\mathbf{P}$ yields its underlying poset $\mathbf{P}^\circ = (X_1 \cup \cdots \cup X_m, \preceq)$ by ignoring the partition. The order dimension of $\mathbf{P}$ is then defined to be that of $\mathbf{P}^\circ$.

Note that if $\mathbf{P} = (X, \preceq)$ is a poset with $|X| = n$, then the number of ways $\mathbf{P}$ can be made into a n-partite poset is the number of linear extensions of $\mathbf{P}$. If $n' > n$, then $\mathbf{P}$ cannot be made into a n'-partite poset.

If $\mathbf{P} = (X, \preceq)$ is a graded poset with a surjective grading function $g : X \to [m]$, then $\mathbf{P}$ can naturally be made into an m-partite poset $\mathbf{P}' = (X_1, \ldots, X_m; \preceq)$ where $X_i = g^{-1}(i)$ for each $i \in [m]$.

Slightly more generally, if a poset $\mathbf{P} = (X, \preceq)$ has a partition $X = X_1 \cup \cdots \cup X_m$ such that $\preceq$ is induced by a collection of bipartite posets $\mathbf{P}_i = (X_i, X_{i+1}; \preceq_i)$ for each $i \in [m-1]$, then $\mathbf{P}$ is also an m-partite poset. Here we have $x \prec y$ in $\mathbf{P}$ iff there is a sequence $x = x_i \prec x_{i+1} \prec \cdots \prec x_{j-1} \prec x_j = y$, where each $x_k \in X_k$ and $x_k \prec_k x_{k+1}$.

In fact, for any poset $\mathbf{P} = (X, \preceq)$ the rank function (mapping each element to the length of the longest chain with that element as its largest element) will make $\mathbf{P}$ into an m-partite poset for some m. Here $m' = m + 1$, the height of $\mathbf{P}$, is the smallest possible m' such that $\mathbf{P}$ can be made into an m'-partite poset.

Remarks: (i) Note that the three examples above increase slightly in generality. (ii) Many authors confuse the notion of a grading with that of a rank. This distinction is made clear in the excellent book by Bernd S. W. Schröder [13].

Example: Consider the poset $\mathbf{P} = (X, \preceq)$ with $X = \{a, b, c, d, e\}$ where $\preceq$ is induced by the two chains $d \prec c \prec b \prec a$ and $d \prec e \prec a$. From $\mathbf{P}$ we can obtain two 4-partite posets $\mathbf{P}' = (\{d\}, \{c, e\}, \{b\}, \{a\}; \preceq)$ and $\mathbf{P}'' = (\{d\}, \{c\}, \{b, e\}, \{a\}; \preceq)$ with $\mathbf{P}$ as their underlying poset. Here $\mathbf{P}'$ is given by the partition that the rank function yields. Further, $\mathbf{P}$ has no grading function and is therefore not a graded poset.

Let $\mathbf{P} = (X_1, \ldots, X_m; \preceq)$ be an m-partite poset and $\mathbf{P}_{i,j}$ be the bipartite sub-poset of $\mathbf{P}$ induced by $X_i \cup X_j$ for each $i < j$ with $i, j \in [m]$. As we saw in the example preceding Observation 3.1, we cannot hope to express $\dim(\mathbf{P})$ in terms of the $\dim(\mathbf{P}_{i,i+1})$'s for $i \in [m-1]$, the order dimensions of these consecutive layers in $\mathbf{P}$. More is needed.

For each $i, j \in [m]$ with $i < j$ let $d_{i,j} = \dim(\mathbf{P}_{i,j})$ and $\mathcal{L}_{i,j}$ be a collection of $d_{i,j}$ linear orders on $X_i \cup X_j$ realizing $\mathbf{P}_{i,j}$. By Lemma 3.2 there is a set $\mathcal{L}^*_{i,j}$ of $d_{i,j}$ linear orders extending $\mathbf{P}$ and each linear order in $\mathcal{L}_{i,j}$. By considering both cases of $x \parallel y$, where $x, y \in X_i$ for some i on one hand, and $x \in X_i$, $y \in X_j$ for some $i \neq j$ on the other, we can see that $\mathcal{R} = \bigcup_{i<j} \mathcal{L}^*_{i,j}$ realizes $\mathbf{P}$. This shows that we can bound $\dim(\mathbf{P})$ in terms of the $\dim(\mathbf{P}_{i,j})$'s. We summarize in the following.

Observation 3.4 *For a multipartite poset* $\mathbf{P} = (X_1, \ldots, X_m; \preceq)$ *we have*

$$\dim(\mathbf{P}) \leq \sum_{i<j} \dim(\mathbf{P}_{i,j}).$$

For an m-partite poset $\mathbf{P}$ let $B(\mathbf{P}) = \max_{i<j}\{\dim(\mathbf{P}_{i,j})\}$. Since there are $\binom{m}{2} = m(m-1)/2$ posets $\mathbf{P}_{i,j}$ we obtain

$$B(\mathbf{P}) \leq \dim(\mathbf{P}) \leq \frac{m(m-1)}{2} B(\mathbf{P}),$$

and hence for a fixed m, we have $\dim(\mathbf{P}) = \Theta(B(\mathbf{P}))$. This can be reduced by a factor of $1/2$ in the following theorem.

Theorem 3.5 *For a multipartite poset* $\mathbf{P} = (X_1, \ldots, X_m; \preceq)$ *we have*

$$B(\mathbf{P}) \leq \dim(\mathbf{P}) \leq \frac{(m-1)(m+3)}{4} B(\mathbf{P}).$$

Proof. Note that if $i_1 < j_1 < i_2 < j_2 < \cdots < i_\ell < j_\ell$ are indices from $[m]$ and $\mathbf{L}_k$ is a linear extension of $\mathbf{P}_{i_k,j_k}$, then a linear extension of $\mathbf{P}$ that includes $\mathbf{L}_1 \prec \mathbf{L}_2 \prec \cdots \prec \mathbf{L}_\ell$ extends $\mathbf{P}$ and each of the $\mathbf{L}_k$. In this way we can find $2 \cdot B(\mathbf{P})$ linear orders extending $\mathbf{P}$ and each $\mathbf{L}_{i,j} \in \mathcal{L}_{i,j}$, where $i + 1 = j$. In general, for each $k \leq \lfloor (m+1)/2 \rfloor$ there are $k \cdot B(\mathbf{P})$ linear orders extending $\mathbf{P}$ and each $\mathbf{L}_{i,j}$, where $i + k - 1 = j$. There are however $1 + 2 + \cdots + (m - \lfloor (m+1)/2 \rfloor)$ ways of choosing a pair $i < j$ with $j - i \geq \lfloor (m+1)/2 \rfloor$. Therefore the total number of linear orders extending $\mathbf{P}$ and each $\mathbf{L}_{i,j} \in \mathcal{L}_{i,j}$ for all $i < j$, will not exceed

$$\begin{aligned}\left[\left(2 + 3 + \cdots + \left\lfloor \frac{m+1}{2} \right\rfloor\right) + \left(1 + 2 + \cdots + \left(m - \left\lfloor \frac{m+1}{2} \right\rfloor\right)\right)\right] \cdot B(\mathbf{P}) \\ \leq \frac{(m-1)(m+3)}{4} B(\mathbf{P}).\end{aligned}$$

Hence we have the theorem. $\square$

To better understand the asymptotic behavior of $\dim(\mathbf{P})$ of an m-partite poset $\mathbf{P}$, define $f(m)$ for each $m \geq 2$ by

$$f(m) = \sup_{\mathbf{P}} \left\{ \frac{\dim(\mathbf{P})}{B(\mathbf{P})} \right\}, \tag{2}$$

where the supremum is taken over all m-partite posets $\mathbf{P}$. By Theorem 3.5 we therefore have that $f(m) \leq (m-1)(m+3)/4$.

For the lower bound of $f(m)$, we start with the following lemma.

Lemma 3.6 *Let $g, h, k \in \mathbb{N}$ with $h, k \geq 2$ and $g \leq \min\{h, k\}$. Let $M \subseteq [h] \times [k]$ be any matching of size g between the columns and rows of $[h] \times [k]$. For disjoint sets $X = \{x_1, \ldots, x_h\}$ and $Y = \{y_1, \ldots, y_k\}$ let $\mathbf{C}_{-g}(h, k)$ be the poset on $X \cup Y$ given by $x_i \prec y_j$ for all $(i, j) \in [h] \times [k] \setminus M$. Then* $\dim(\mathbf{C}_{-g}(h, k)) = \max\{2, g\}$.

Proof. Assume $g \geq 2$. By a suitable permutation we may assume that $M = \{(1, 1), \ldots, (g, g)\}$. Since the poset induced by $\{x_1, \ldots, x_g\} \cup \{y_1, \ldots, y_g\}$ is the standard example $\mathbf{S}_{2g}$ we have that $\dim(\mathbf{C}_{-g}(h, k)) \geq g$.

Let L_x denote the linear order $x_1 \prec x_3 \prec x_4 \prec \cdots \prec x_{h-1} \prec x_h \prec x_2$ and similarly let L_y denote $y_1 \prec y_3 \prec y_4 \prec \cdots \prec y_{k-1} \prec y_k \prec y_2$. If $i \in [h]$ then $L_x(\hat{i})$ denotes the linear order obtained from L_x by removing x_i and similarly for $L_y(\hat{j})$. For any linear order L let L^{op} denote the opposite, or reverse, linear order of L. In this case $\mathbf{C}_{-g}(h, k)$ is realized by the following g linear orders

$$\begin{aligned} &L_x(\hat{1})^{\mathrm{op}} \prec y_1 \prec x_1 \prec L_y(\hat{1}), \\ &L_x(\hat{\ell}) \prec y_\ell \prec x_\ell \prec L_y(\hat{\ell})^{\mathrm{op}} \quad \text{for } \ell \in \{2, \ldots, g\}. \end{aligned}$$

Hence $\dim(\mathbf{C}_{-g}(h, k)) \leq g$. The case $g = 1$ gives in similar fashion $\dim(\mathbf{C}_{-1}(h, k)) = 2$. □

Note that $\mathbf{C}_{-g}(h, k))$ is the complete bipartite poset on X and Y except for the g relations $x_i \prec y_j$ where $(i, j) \in M$.

Theorem 3.7 *For $m \geq 2$ we have that $f(m)$ defined in (2) satisfies*

$$f(m) \geq \frac{m^2 - 1}{4}.$$

Proof. For $d, h, k \geq 2$ let $A = \{x_{i,j} : (i, j) \in [dh] \times [dk]\}$ and $B = \{y_{i,j} : (i, j) \in [dh] \times [dk]\}$ be two disjoint sets of d^2hk elements each. Let $\mathbf{P} = (A \cup B; \preceq)$ be given by

$$x_{i_1,j_1} \prec y_{i_2,j_2} \Leftrightarrow (i_1, j_1) \neq (i_2, j_2).$$

Here $\mathbf{P}$ is the standard example on $2d^2hk$ elements so $\dim(\mathbf{P}) = d^2hk$. Let $X_1, \ldots, X_h, Y_1, \ldots, Y_k$ be given by $X_p = \{x_{i,j} : (i, j) \in \{(p-1)d+1, \ldots, pd\} \times [dk]\}$ for each $p \in [h]$ and $Y_q = \{y_{i,j} : (i, j) \in [dh] \times \{(q-1)d+1, \ldots, qd\}\}$ for each $q \in [k]$. This partition of $A \cup B$ makes $\mathbf{P}$ into a $(h+k)$-partite poset $(X_1, \ldots, X_h, Y_1, \ldots, Y_k; \preceq)$. We note that each of $X_p \cup X_q$ and $Y_p \cup Y_q$ is an antichain in $\mathbf{P}$ of order dimension two. Since the sub-poset of $\mathbf{P}$ induced by $X_p \cup Y_q$ is $\mathbf{C}_{-d^2}(d^2k, d^2h)$ we have by Lemma 3.6 that $B(\mathbf{P}) = d^2$. Hence we have

$$f(h + k) \geq \frac{\dim(\mathbf{P})}{B(\mathbf{P})} = \frac{d^2hk}{d^2} = hk.$$

Putting $(h, k) = (n, n)$ on one hand and $(h, k) = (n, n + 1)$ on the other yields a lower bound for $f(m)$ both for even and odd m. Hence, we have the theorem. □

Note that the example provided in the above proof of Theorem 3.7 shows that both $\dim(\mathbf{P})$ and $B(\mathbf{P})$ can be arbitrarily large.

By Theorems 3.5 and 3.7 we have the following.

Corollary 3.8 *If $f(m)$ is the function from (2) then for all $m \geq 2$ we have*

$$\frac{m^2 - 1}{4} \leq f(m) \leq \frac{(m-1)(m+3)}{4}.$$

By Corollary 3.8 we have $\lim_{m\to\infty} \frac{f(m)}{m^2} = \frac{1}{4}$, so the bounds given in Corollary 3.8 are asymptotically tight.

By imposing some additional conditions on the multipartite partite poset, conditions that many familiar graded posets satisfy, we can obtain much better bounds on the order dimension than obtained in Theorem 3.5.

Definition 3.9 *Let $\mathbf{P} = (X_1, \ldots, X_m; \preceq)$ be an m-partite poset. We say that $\mathbf{P}$ has the* incomparable cover property (ICCP) *if for every incomparable pair $x \parallel y$ of $\mathbf{P}$, there is an $x' \in X_1$ and $y' \in X_m$ such that (1) $x' \leq x$, (2) $y' \geq y$ and (3) $x' \parallel y'$.*

Note that x' and y' are necessarily minimal and maximal elements of $\mathbf{P}$ respectively.

Theorem 3.10 *If an m-partite poset $\mathbf{P}$ has the ICCP, then*

$$\dim(\mathbf{P}) = B(\mathbf{P}) = \dim(\mathbf{P}_{1,m}).$$

Proof. Let $\dim(\mathbf{P}_{1,m}) = d_{1,m}$ and $\mathcal{L}_{1,m}$ be a realizer of $\mathbf{P}_{1,m}$ containing $d_{1,m}$ linear orders on $X_1 \cup X_m$. By Lemma 3.2 there is a set $\mathcal{L}^*_{1,m}$ of $d_{1,m}$ linear orders extending $\mathbf{P}$ and each linear order in $\mathcal{L}_{1,m}$. Restricting each linear order in $\mathcal{L}^*_{1,m}$ to $X_i \cup X_j$ we obtain a collection of $d_{1,m}$ linear orders, which by the ICCP is a realizer of $\mathbf{P}_{i,j}$ for each $i < j$. Hence we have the theorem. □

EXAMPLE: Consider a graded poset $\mathbf{P} = (X, \preceq)$ with a grading function $g : X \to [4]$. If $X_i = g^{-1}(i)$, we have a partition $X = X_1 \cup X_2 \cup X_3 \cup X_4$ making $\mathbf{P}$ into a 4-partite poset. If $\mathbf{P}$ has the ICCP, then by the above Theorem 3.10 and Hiraguchi's theorem (from the remark right after Observation 2.2) we have

$$\frac{|X_1| + |X_4|}{2} \geq \dim(\mathbf{P}_{1,4}) = \dim(\mathbf{P}) \geq \dim(\mathbf{P}_{2,3}).$$

Hence, any graded poset $\mathbf{P}$ with $2\dim(\mathbf{P}_{2,3}) > |X_1| + |X_4|$ does not satisfy the ICCP.

For an n-dimensional polytope P let $\mathcal{F}_P$ be its face lattice. For an integer vector $\tilde{k} = (k_1, \ldots, k_m)$ with $1 \leq k_1 < k_2 < \cdots < k_m < n$, let $\mathcal{F}_P(\tilde{k})$ denote the face sub-poset of $\mathcal{F}_P$ induced by the k_i-dimensional faces of P for all $i \in [m]$. With this setup we have the following.

Theorem 3.11 *For every n-dimensional polytope P and each $\tilde{k}$ with $1 \leq k_1 < k_2 < \cdots < k_m < n$, then $\mathcal{F}_P(\tilde{k})$ has the ICCP.*

Proof. Let $H_1, \ldots, H_p$ be the hyperplanes in $\mathbb{R}^n$ that bound P. Note that each k-face F_k of P is determined by exactly $n-k$ of the hyperplanes $F_k = P \cap H_{i_1} \cap \cdots \cap H_{i_{n-k}}$.

Let F' and F'' be two incomparable faces of P with $\dim(F'), \dim(F'') \in \{k_1, \ldots, k_m\}$. Then there is a 0-face $F_0 = \{u\}$ such that $u \in F' \setminus F''$. Let $F'' = P \cap H_{i_1} \cap \cdots \cap H_{i_{n-k}}$ where $k = \dim(F'')$. Since $u \notin P \cap H_{i_1} \cap \cdots \cap H_{i_{n-k}}$ there must be a hyperplane, say H_{i_1} that does not contain the point (i.e. vertex) u. In this case we have $u \notin F_{n-1} := P \cap H_{i_1}$ so $F_0 \subseteq F'$ and $F'' \subseteq F_{n-1}$ where F_0 and F_{n-1} are incomparable in $\mathcal{F}_P(\tilde{k})$. Therefore $\mathcal{F}_P(\tilde{k})$ has the ICCP. □

For integers $1 \leq k_1 < k_2 < \cdots < k_m < n$ denote by $\mathbf{P}(k_1, \ldots, k_m; n)$ the sub-poset of $\mathbb{P}([n])$ that consists of the union of all k_i-element subsets of $[n]$ where $i \in [m]$. Since $\mathbb{P}([n])$ is the face lattice of the standard n-simplex we have by Theorem 3.11 the following.

Corollary 3.12 *$\mathbf{P}(k_1, \ldots, k_m; n)$ has the ICCP.*

By Theorem 3.10 and Corollary 3.12 it is immediate that $\dim(\mathbf{P}(1, \ldots, n-1; n)) = \dim(\mathbf{P}(1, n-1; n)) = n$, since $\mathbf{P}(1, n-1; n)$ is the standard example $\mathbf{S}_{2n}$. Note that here we avoided the use of Dilworth's Chain Decomposition Theorem (product version) which is the standard way to show that $\dim(\mathbf{P}(1, \ldots, n-1; n)) = n$ [13, p. 171,232].

For a poset $\mathbf{P} = (X, \preceq)$ let $\Delta(\mathbf{P}) = \max_{x \in X} |\{y \in X : y \preceq x \text{ or } y \succeq x\}|$, that is, the maximum number of elements that are comparable to a single element. A celebrated result in [6] states that $\dim(\mathbf{P}) \leq 50\Delta(\mathbf{P})(\log \Delta(\mathbf{P}))^2$.

If $1 \leq k_1 < k_2 < n$, we have that in the poset $\mathbf{P}(k_1, k_2; n)$ each maximal element (i.e. a k_2-set) is comparable to exactly $\binom{k_2}{k_1} + 1$ other elements and each minimal element (i.e. a k_1-set) is comparable to exactly $\binom{n-k_1}{k_2-k_1} + 1$ elements. Also, note that the function $f : [n] \to \mathbb{N}$ given by $f(k) = k!(n-k)!$ satisfies $f(k) = f(n-k)$ and is decreasing for the first half of the numbers k among $[n]$ and increasing for the second half. From this we see that $\Delta(\mathbf{P}(k_1, k_2; n)) = \binom{k_2}{k_1} + 1$ if, and only if, $k_1 + k_2 \leq n$. Hence we have

$$\Delta(k_1, k_2) := \Delta(\mathbf{P}(k_1, k_2; n)) = \begin{cases} \binom{k_2}{k_1} + 1 & \text{if } k_1 + k_2 \leq n, \\ \binom{n-k_1}{k_2-k_1} + 1 & \text{if } k_1 + k_2 \geq n. \end{cases} \tag{3}$$

In particular, by Theorem 3.10, Corollary 3.12 and the above we get that

$$\dim(k_1, \ldots, k_m; n) = \dim(k_1, k_m; n) \leq \Delta(k_1, k_m)(\log \Delta(k_1, k_m))^2, \tag{4}$$

where $\Delta(k_1, k_m)$ is defined as in (3). Hence, the *mere reason* that we obtain an upper bound in (4) solely as a function of $\Delta(k_1, k_m)$ instead of $\Delta(\mathbf{P}(k_1, \ldots, k_m; n))$, is given by Theorem 3.11.

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