On the extension of vertex maps to graph homomorphisms

Geir Agnarsson [*]   *Li Chen* [†]

Abstract

A *reflexive graph* is a simple undirected graph where a loop has been added at each vertex. If G and H are reflexive graphs and $U \subseteq V(H)$, then a vertex map $f : U \to V(G)$ is called *nonexpansive* if for every two vertices $x, y \in U$, the distance between $f(x)$ and $f(y)$ in G is at most that between x and y in H. A reflexive graph G is said to have the *extension property (EP)* if for every reflexive graph H, every $U \subseteq V(H)$ and every nonexpansive vertex map $f : U \to V(G)$, there is a graph homomorphism $\phi_f : H \to G$ that agrees with f on U. Characterizations of EP-graphs are well known in the mathematics and computer science literature. In this article we determine when exactly, for a given "sink"-vertex $s \in V(G)$, we can obtain such an extension $\phi_{f;s}$ that maps each vertex of H closest to the vertex s among all such existing homomorphisms ϕ_f. A reflexive graph G satisfying this is then said to have the *sink extension property (SEP).* We then characterize the reflexive graphs with the *unique sink extension property (USEP)*, where each such sink extensions $\phi_{f;s}$ is unique.

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1 Introduction

The problem of determining whether or not vertex maps can be extended to graph homomorphisms is quite natural and has been discussed extensively in both the mathematics and computer science literature. The problems addressed and solved in this article were inspired by some partial results from [4], [5] and [6]. Although not written in the language of graphs and their homomorphisms, the main results of [4], [5] and the parts of [6] that are relevant to this article, turn out to be special cases of celebrated results from [13] and [8]. These results can also be found in the recent excellent books [12] and [14]. To the best of the author's knowledge, the problems solved in this article, to characterize SEP-graphs and USEP-graphs (see Definitions 3.1 and 3.2 in Section 3 below), have not been discussed elsewhere. We will in this article for the most part use the notation and names from [12] for the sake of consistency.

The study of extending vertex maps to graph homomorphisms is inseparable from that of retracts of graphs (see definition in the next Section 2). We will now briefly discuss some highlights of this ongoing study of graph retracts:

It is generally believed that graph retracts were first studied in a comprehensive way in the 70s first by Pavol Hell [10], [11], then a little later by Ivan Rival and Richard Nowakowski [18], [19].

[*]Department of Mathematical Sciences, George Mason University, MS 3F2, 4400 University Drive, Fairfax, VA – 22030, USA, `geir@math.gmu.edu`

[†]Department of Electrical Engineering and Computer Science, University of the District of Columbia, 4200 Connecticut Avenue NW, Washington, DC – 20008, USA, `lchen@udc.edu`

Some highlights of later contributions in this discussion includes that of [13] a paper from which the discussion in [12] on retracts is partly based on, [15] where retracts from a metric point of view is discussed, [1] on retracts in bipartite graphs and [16], where the "holes" are discussed. There, a hole is precisely what Helly graphs do not have (see definition of the Helly property in the next Section 2). When exactly cycles are retracts of planar graphs is determined in [27], and in [26] the Helly property is used on posets. A game theoretic approach involving cops and robbers is taken in [20]. Further studies of absolute retracts from a chromatic point of view is presented in [24]. Retracts of posets is studied in [7] and [8]. Further studies of retracts are given by Erwin Pesch in [21], [22] and [23]. The more computational aspect of determining when a given graph is a retract is given in [3] and [9]. In [25] the retract is studied from the more classical graph-contraction point of view.

The rest of this article is organized as follows: In Section 2 we introduce our notation, basic definitions and recapitulate some known relevant results. Sections 3 and 4 contain the main results of this article, namely Corollary 3.14 and Theorem 4.6. More specifically, in Section 3 we show that we can extend a vertex map to a homomorphism that further maps everything as close to a given sink-vertex as possible. Finally, in Section 4 we determine for which graphs the homomorphism discussed in Section 3 is unique.

2 Definitions, notation and recap of related results

Here in this section, we present our notation, terminology, and basic definitions. We also state and recapitulate some known results closely related to what we consider and use in following sections. There are numerous nice published directly related results, but since the notation is non-standard it can be difficult to parse through and clearly see what implies what. Hence, it seems worthwhile to briefly discuss these results here in this section in terms of the notation that we will be using. Most of the following terminology definitions and results can be found in [12] and also some in [14]. In addition, we refer to [12, 2.13 Remarks] and [14, 5.6 Notes] for more detailed discussion on the history and origin of the results presented here in this section. We start with the notation and basic definitions:

The natural numbers $\{1, 2, \ldots\}$ will be denoted by $\mathbb{N}$ and for $k \in \mathbb{N}$ the set $\{1, \ldots, k\}$ will be denoted by $[k]$. For a set S, the set of all k-element subsets of S will be denoted by $\binom{S}{k}$. A *simple graph* G is an ordered pair $(V(G), E(G))$ where $V(G)$ is a finite set of vertices and $E(G) \subseteq \binom{V(G)}{2}$ is the set of edges of G. A *reflexive graph* is a simple graph G where we have added a loop at each vertex. Hence, the edge set $E(G)$ of a reflexive graph can be viewed as a subset of the disjoint union $\binom{V(G)}{1} \cup \binom{V(G)}{2}$ that contains $\binom{V(G)}{1}$. In this article a *graph* is always either simple or reflexive. The *distance* between vertices x and y in G, denoted by $d_G(x, y)$, is the minimum number of edges in a path between x and y in G. A *homomorphism* $\phi : H \to G$ is a tuple $\phi = (\phi_V, \phi_E)$ where $\phi_V : V(H) \to V(G)$ and $\phi_E : E(H) \to E(G)$ are maps that satisfy the compatibility condition $\phi_E(\{x, y\}) = \{\phi_V(x), \phi_V(y)\}$. Note that the edge map ϕ_E is completely determined by vertex map ϕ_V. When there is no danger of ambiguity we will write $\phi(x)$ instead of $\phi_V(x)$ for a vertex x and similarly for an edge. Note also that for any $x, y \in V(H)$ we have $d_G(\phi(x), \phi(y)) \le d_H(x, y)$. Let H and G be graphs and $U \subseteq V(H)$. A vertex map $f : U \to V(G)$ is *nonexpansive (NE)* if for all $x, y \in U$ we have $d_G(f(x), f(y)) \le d_H(x, y)$. Note that if H and G are reflexive graphs then, $\phi : H \to G$ is a homomorphism if, and only if, $\phi_V : V(H) \to V(G)$ is an NE-map. For simple graphs H and G however, a vertex map $\theta : V(H) \to V(G)$ is an NE-map if, and only if, we have the following.

$$\text{For all } x, y \in V(H) : \{x, y\} \in E(H) \Rightarrow \{\theta(x), \theta(y)\} \in E(G) \text{ or } \theta(x) = \theta(y). \tag{1}$$

Maps satisfying (1) have been called *weak homomorphisms* of simple graphs in the literature (see [14].) Note that if we add a loop at each vertex of the graph G, thereby obtaining the reflexive graph G', a weak homomorphism $\theta : H \to G$ is equivalent to a homomorphism $\theta' : H \to G'$.

Definition 2.1 *A reflexive graph G has the* extension property (EP) *if for every reflexive graph H, every $U \subseteq V(H)$ and every NE-map $f : U \to V(G)$ there is a homomorphism $\phi_f : H \to G$ that agrees with f on U.*

It should be clear how the EP can be modified for simple graphs. In [14, p. 152] it is shown that a simple graph with the Helly property (see definition here below) has the EP. Further, a complete characterization of such simple Helly-graphs is also given there. For a graph G, a *quotient graph* of G, denoted by G/σ, is a graph with σ as vertex set, where σ is a set partition of $V(G)$, and where $\{S, S'\}$ is an edge in G/σ iff there is an edge $\{x, y\} \in E(G)$ with $x \in S$ and $y \in S'$. For each quotient G/σ we have a natural surjective homomorphism $p : G \to G/\sigma$. A graph G is a *retract* of H if G is a subgraph of H and there is a homomorphism $r : H \to G$ the restriction of which to $G \subseteq H$ is the identity homomorphism of G. Note that if $r : H \to G$ is a retraction then, for every $x, y \in G$ we have $d_G(x, y) = d_G(r(x), r(y)) \leq d_H(x, y)$, but since trivially $d_H(x, y) \leq d_G(x, y)$ for any subgraph G of H, we have here that

$$d_G(x, y) = d_H(x, y) \text{ for any } x, y \in V(G). \tag{2}$$

A subgraph G of H satisfying (2) is called *isometric*. A graph G is an *absolute retract* if G is a retract of every graph H in which it is an isometric subgraph of. For a graph G and a nonnegative integer ℓ, a set $N_\ell^G[x] = \{z \in V(G) : d_G(x, z) \leq \ell\}$ is called a *closed ball* or a *closed neighborhood* of G. A graph G is said to have the *Helly property* if every collection of closed balls in G, the intersection of each pair of which is nonempty, has a nonempty intersection. The following nice characterization stems from [13] and is given in [12].

Theorem 2.2 *A reflexive graph is an absolute retract if, and only if, it has the Helly property.*

In view of the above Definition 2.1 we have in fact the following:

Observation 2.3 *For a reflexive graph G the following are equivalent:*

1. *G has the EP.*
2. *G is an absolute retract.*
3. *G has the Helly property.*

Although the equivalence in Observation 2.3 is known and can be obtained by a combination of theorems and such from [14] and [12], we conclude this section however by a simple, algebraic and self contained proof of the missing part of Theorem 2.2, that a reflexive graph has the EP if, and only if, it is an absolute retract:

Proof. (Obs. 2.3) First assume that G is an isometric subgraph of H and that G has the EP. In this case the identity map $V(G) \to V(G)$ is an NE-map which, by the EP of G, can be extended to a homomorphism $r : H \to G$, showing that G is a retract of H.

Conversely, for reflexive graphs H and G, a vertex set $U \subseteq V(H)$ and a vertex map $f : U \to V(G)$, let $(H+G)/f$ denote the quotient of the disjoint union of H and G where we have for each $x \in U$ identified $f(x)$ with x. More specifically, in the partition of the disjoint union of $V(G)$ and $V(H)$ that the map f induces, each $x \in V(H) \setminus U$ and each $y \in V(G) \setminus f(U)$ is a singleton set in the partition, whereas for each $y \in f(U)$ the set $S_y = \{y\} \cup f^{-1}(y)$ makes up a non-singleton set of the partition.

Lemma 2.4 *If $f : U \to V(G)$ is an NE-map, then $(H+G)/f$ contains G as an isometric subgraph.*

Proof. It is clear that $(H + G)/f$ contains G as a subgraph. Let $x, y \in V(G)$ be given and $P = (x_0, x_1, \ldots, x_k)$ be a shortest x,y-path in $(H + G)/f$ that has the fewest possible edges from $E(H) \setminus E(G)$. If P contains no edge in $E(H)$ then $k = d_G(x, y)$ and we are done. Otherwise, there is a sub-path $P' = (x_i, \ldots, x_j)$ of P with $i < j$ and $x_i, x_j \in U$ containing an edge from $E(H) \setminus E(G)$ that lies entirely in the image of H in $(H + G)/f$. Since f is an NE-map there is however another path P'' starting at x_i and ending at x_j with length at most that of P' and that lies in the image of G in $(H+G)/f$. Replacing P' by P'' in P we obtain a x,y-path in $(H+G)/f$ of length at most k, but with fewer edges from $E(H) \setminus E(G)$. This contradicts our choice of P and completes the argument. ∎

Assume now that G is an absolute retract. Let H be a reflexive graph, $U \subseteq V(H)$ and $f : U \to V(G)$ a NE-map. By Lemma 2.4 there is retraction $r : (H + G)/f \to G$. In this case $\phi_f : H \to G$ defined by

$$H \xrightarrow{p} (H + G)/f \xrightarrow{r} G,$$

(where p denotes the natural projection), agrees with f on U. This completes our proof. ∎

Although a priori a different and seemingly a more restrictive definition, graphs with the EP turn out to be precisely the absolute retracts. In the next sections we will continue in this manner by studying two sub-classes of reflexive graphs satisfying the EP and describing them completely.

3 On graphs satisfying the SEP or the USEP

In this section we discuss two stronger conditions than the EP. We study whether or not we can impose the additional condition on our extending homomorphism that it maps all the vertices closest to a given sink-vertex among all existing extensions of our given NE-map, and whether or not such an extension homomorphism is unique.

For reflexive graphs G and H and $s \in V(G)$ there is always a homomorphism $\phi_s : H \to G$ such that for every homomorphism $\phi : H \to G$ and every vertex $x \in V(H)$ we have $d_G(\phi_s(x), s) \leq d_G(\phi(x), s)$, namely by letting $\phi_s(x) = s$ for each $x \in V(H)$. However, if we require that both ϕ and ϕ_s extend a given NE-map $f : U \to V(G)$ for some $U \subseteq V(H)$, then such a trivial choice for our ϕ_s is clearly not an option.

Definition 3.1 *Let G be a reflexive graph with the EP and $s \in V(G)$. For H a reflexive graph, $U \subseteq V(H)$ and $f : U \to V(G)$ an NE-map, an extending homomorphism $\phi_{f;s} : H \to G$ of f*

is called a s-sink extension (s-SE) *if* $d_G(\phi_{f;s}(x), s) \leq d_G(\phi_f(x), s)$ *for all* $x \in V(H)$ *and for all homomorphisms* ϕ_f *extending* f*. If every* f *can be extended to an* s*-SE, then* G *has the* s-sink extension property (s-SEP)*. If* G *has the* s*-SEP for every* $s \in V(G)$ *then* G *has the* sink extension property (SEP)*.*

Note that if G has the SEP and $s \in V(G)$, H, $U \subseteq V(H)$ and NE-map $f : U \to V(G)$ are given, then for each $x \in V(H)$ the nonnegative integer $m(x) = \min_f\{d_G(\phi_f(x), s)\}$ is unique, namely $d_G(\phi_{f;s}(x), s)$ where $\phi_{f;s} : H \to G$ is an s-SE of f. However, the homomorphism $\phi_{f;s}$ itself is not necessarily unique.

Definition 3.2 *Let* G *be a reflexive graph. We say that* G *has the* unique sink extension property (USEP) *if* G *has the SEP and for every* $s \in V(G)$*, every reflexive graph* H*, every* $U \subseteq V(H)$ *and every NE-map* $f : U \to V(G)$*, the* s*-SE* $\phi_{f;s} : H \to G$ *is unique.*

Recall that a *block* of a graph is a connected maximal subgraph with no cut-vertices. Call a connected reflexive graph G a *block-tree*, if each block on two or more vertices of G is a clique. The following claims are easy to prove by induction on the number of blocks.

Claim 3.3 *Between any two vertices in a block-tree there is a unique shortest path.*

If G is a block-tree and $s \in V(G)$, let T_s be the subgraph of G formed by the union of all the shortest paths between s and x for each $x \in V(G)$.

Claim 3.4 *For a block-tree* G *and* $s \in V(G)$ *we have the following:*

1. T_s *is a spanning subtree of* G*.*
2. *Viewing* T_s *as a rooted tree at* s*, then the vertices of each block in* G *form a rooted sub-star in* T_s *with all its vertices, except the star-root, on the level just below the star-root.*
3. *For any vertices* $x, y \in V(G)$ *we have* $d_{T_s}(x, y) \leq d_G(x, y) + 1$*.*

We can now prove the following.

Theorem 3.5 *Every reflexive block-tree has the USEP.*

Proof. Let G be a reflexive block-tree and $s \in V(G)$. View T_s as rooted spanning tree at s of G. Let H be a reflexive graph, $U \subseteq V(H)$ and $f : U \to V(G)$ an NE-map. For each $x \in V(H)$ and $u \in U$ let

$$N_u(x) = \{v \in V(G) : d_G(f(u), v) \leq d_H(u, x)\}.$$

There is a unique vertex $z_u(x) \in N_u(x)$ that is closest to s among all vertices of $N_u(x)$, namely the vertex on the unique path from $f(u)$ to s in T_s at distance $\min\{d_H(u, x), d_G(f(u), s)\}$ from $f(u)$ in G. We now argue that the vertices $\{z_u(x) : u \in U\}$ lie on an ancestral path in T_s w.r.t. the root s: Assume that there are $u', u'' \in U$ where neither of $z_{u'}(x)$ or $z_{u''}(x)$ is an ancestor of the other in T_s. This means in particular that neither $z_{u'}(x)$ nor $z_{u''}(x)$ equals s and that $d_{T_s}(z_{u'}(x), z_{u''}(x)) \geq 2$. Hence, by definition of $z_{u'}(x)$ and $z_{u''}(x)$ we then have

$$\begin{aligned} d_{T_s}(f(u'), f(u'')) &= d_{T_s}(f(u'), z_{u'}(x)) + d_{T_s}(z_{u'}(x), z_{u''}(x)) + d_{T_s}(z_{u''}(x), f(u'')) \\ &\geq d_H(u', x) + 2 + d_H(u'', x) \\ &\geq d_H(u', u'') + 2. \end{aligned}$$

By Claim 3.4 we then have $d_G(f(u'), f(u'')) > d_H(u', u'')$ which contradicts that f is an NE-map. Therefore the vertices of $\{z_u(x) : u \in U\}$ all lie on an ancestral path in T_s rooted at s.

Define $\phi_{f;s}(x)$ to be the unique vertex $z^*(x) \in \{z_u(x) : u \in U\}$ that is the descendant of all of them (note, here a vertex is considered a descendant of itself!) Since for each $u \in U$ we have $N_u(u) = \{f(u)\}$, we clearly have $z^*(u) = f(u)$ and hence $\phi_{f;s}$ agrees with f on U.

If now x' is adjacent to x in H, then $|d_H(u,x) - d_H(u,x')| \in \{0,1\}$ for each $u \in U$ and therefore each of $z_u(x')$ is adjacent to or equal to $z_u(x)$ in G. This in return implies that $z^*(x)$ is also adjacent to or is equal to $z^*(x')$ in G, and hence $d_G(z^*(x), z^*(x')) \leq 1$, showing that $\phi_{f,s} : H \to G$ is a homomorphism. By the mere definition of $\phi_{f;s}$, it is clear that $\phi_{f;s}$ is an s-SE. Finally, by the uniqueness of $z^*(x)$ we see that $\phi_{f;s}$ is uniquely determined. Since $s \in V(G)$ was arbitrary we have completed the proof. $\square$

NOTE: Similar arguments as used in the above proof can be used to show directly that every block-tree has the Helly property: Assuming this property, since $f : U \to V(G)$ is an NE-map we have $N_u(x) \cap N_{u'}(x) \neq \emptyset$ for every $u, u' \in U$ and hence by the Helly property of $\{N_u(x) : u \in U\}$ we have

$$N(x) = \bigcap_{u \in U} N_u(x) \neq \emptyset.$$

By induction on $|U|$ we can see that $N(x)$ is a block-subtree of G, and hence contains a unique vertex closest to s among all vertices of $N(x)$. This vertex turns out to be precisely the vertex $z^*(x)$ defined in the above proof.

REMARK: In the case of a reflexive tree T, the definition of our s-SE $\phi_{f;s}$ in the proof of Theorem 3.5, has a physical interpretation as follows: Let T be rooted at s and direct each edge from a child to its parent, so that all vertices of T except s have out-degree one whereas s has out-degree zero. Call this digraph $\vec{T}_s$ and assume further that each edge is straight and has a unit length. Now, imagine there is a gravitational force field on $\vec{T}_s$ that pulls along its directed edges. Assume likewise that the edges in H also have unit length and that we identify each $x \in U$ with its image $f(x)$ in $\vec{T}_s$ and form the quotient $(H + \vec{T}_s)/f$ (where the edges of T are directed and the ones from H are not.) Keeping $\vec{T}_s$ rigid and letting the vertices of H hang along the gravitational field of $\vec{T}_s$, then each vertex $x \in H$ will place precisely at the unique vertex as close to the root s as H allows. This is exactly the vertex $z^*(x)$ defined in the proof of Theorem 3.5 in the case of reflexive tree T.

Since a reflexive path is in particular a reflexive block-tree, we have by Theorem 3.5 the following trivial but important corollary.

Corollary 3.6 *Every reflexive path has the USEP, and hence the SEP.*

For reflexive graphs G_1 and G_2 their *Cartesian product* or just *product*, denoted by $G_1 \times G_2$, is the reflexive graph with vertex set $V(G_1) \times V(G_2)$ (the usual Cartesian product) and where

$$\{(x_1, x_2), (y_1, y_2)\} \in E(G_1 \times G_2) \Leftrightarrow \{x_1, y_1\} \in E(G_1) \text{ and } \{x_2, y_2\} \in E(G_2).$$

It is easy to see that the product of reflexive graphs is an associative operation and hence it makes sense to talk about the product of k reflexive graphs $\widetilde{G} = G_1 \times \cdots \times G_k$ as the reflexive graph with vertices $V(\widetilde{G}) = V(G_1) \times \cdots \times V(G_k)$ and where for two vertices $\tilde{x} = (x_1, \ldots, x_k)$

and $\tilde{y} = (y_1, \ldots, y_k)$ we have $\{\tilde{x}, \tilde{y}\} \in E(\widetilde{G}) \Leftrightarrow \{x_i, y_i\} \in E(G_i)$ for each $i \in [k]$. Note that for $\tilde{x}, \tilde{y} \in V(\widetilde{G})$ we have

$$d_{\widetilde{G}}(\tilde{x}, \tilde{y}) = \max_{1 \le i \le k}\{d_{G_i}(x_i, y_i)\}. \tag{3}$$

Also, for each $i \in [k]$ the natural projection $\pi_i : \widetilde{G} \to G_i$ is given by $\pi_i(\tilde{x}) = x_i$. It is clear that each π_i is a homomorphism of reflexive graphs. If $k \in \mathbb{N}$ and $\phi_i : H \to G_i$ is a homomorphism for each $i \in [k]$, then $\tilde{\phi} : H \to \widetilde{G}$ given by $\tilde{\phi} = (\phi_1, \ldots, \phi_k)$ is the map $x \mapsto (\phi_1(x), \ldots, \phi_k(x))$ for each $x \in H$. By (3) we note the following:

Observation 3.7 *Let $k \in \mathbb{N}$.*

1. *If $\phi_i : H \to G_i$ is a homomorphism for each $i \in [k]$ then $\tilde{\phi} : H \to \widetilde{G}$ given by $\tilde{\phi} = (\phi_1, \ldots, \phi_k)$ is also a homomorphism.*
2. *Conversely, if $\tilde{\phi} : H \to \widetilde{G}$ is a homomorphism, then $\tilde{\phi}$ has the form $\tilde{\phi} = (\phi_1, \ldots, \phi_k)$ where each homomorphism $\phi_i : H \to G_i$ is a uniquely determined by $\phi_i = \pi_i \circ \tilde{\phi}$.*

A celebrated complete description of absolute retracts is given by the following theorem, the statement of which can be found in [14, Thm. 5.7] and [12, Cor. 2.56] but stems from [8].

Theorem 3.8 *A reflexive graph is an absolute retract if, and only if, it is a retract of product of reflexive paths.*

Assume now $G_1, \ldots, G_k$ are reflexive graphs with the SEP and $\widetilde{G} = G_1 \times \cdots \times G_k$. Let $\tilde{s} = (s_1, \ldots, s_k) \in V(\widetilde{G})$ a vertex, H a reflexive graph, $U \subseteq V(H)$ and $\tilde{f} : U \to V(\widetilde{G})$ an NE-map. If $\pi_i : \widetilde{G} \to G_i$ is the projection onto the i-th coordinate then each $f_i = \pi_i \circ \tilde{f} : U \to V(G_i)$ is also an NE-map for each i, which can be extended to an s_i-SE $\phi_{f_i;s_i} : H \to G_i$. Then $\tilde{\phi}_{\tilde{f};\tilde{s}} = (\phi_{f_1;s_1}, \ldots, \phi_{f_k;s_k}) : H \to \widetilde{G}$, yields a uniquely determined homomorphism. Suppose now that $\tilde{\theta}_{\tilde{f}} : H \to \widetilde{G}$ is a homomorphism that also extends $\tilde{f}$. In this case $\theta_{f_i} : H \to G_i$ extends the NE-map $f_i : U \to G_i$ for each $i \in [k]$. In this case we have $d_{G_i}(\phi_{f_i;s_i}(x), s_i) \le d_{G_i}(\theta_{f_i}(x), s_i)$ for each $i \in [k]$, and hence

$$d_{\widetilde{G}}(\tilde{\phi}_{\tilde{f};\tilde{s}}(x), \tilde{s}) = \max_{1 \le i \le k}\{d_{G_i}(\phi_{f_i;s_i}(x), s_i)\} \le \max_{1 \le i \le k}\{d_{G_i}(\theta_{f_i}(x), s_i)\} = d_{\widetilde{G}}(\tilde{\theta}_{\tilde{f}}(x), \tilde{s}).$$

This shows that $\tilde{\phi}_{\tilde{f};\tilde{s}}$ is a $\tilde{s}$-SE. Since $\tilde{s} \in V(\widetilde{G})$ was arbitrary, we have the following.

Observation 3.9 *A product of reflexive graphs with the SEP has the SEP.*

NOTE: We can not replace "SEP" with "USEP" in Observation 3.9 as the following example will demonstrate.

EXAMPLE: Let H be the reflexive path on three vertices a, b, c in this order. Likewise let P_1 and P_2 be the reflexive paths on the vertices $\{0, 1\}$ and on $\{0, 1, 2\}$ in this order respectively. If $U = \{a, c\}$ and $\tilde{f} : U \to V(P_1 \times P_2)$ is given by $\tilde{f}(a) = (0, 0)$ and $\tilde{f}(c) = (1, 2)$, then $\tilde{f}$ is clearly an NE-map. Assume our sink $\tilde{s} = (0, 0)$. Here there are precisely two legitimate extensions $\tilde{\phi}_{\tilde{f};1}$ and $\tilde{\phi}_{\tilde{f};2}$ where $\tilde{\phi}_{\tilde{f};1}(b) = (0, 1)$ and $\tilde{\phi}_{\tilde{f};2}(b) = (1, 1)$. These homomorphisms satisfy

$$d_{P_1 \times P_2}(\tilde{\phi}_{\tilde{f};1}(x), \tilde{s}) = d_{P_1 \times P_2}(\tilde{\phi}_{\tilde{f};2}(x), \tilde{s})$$

for all $x \in \{a, b, c\}$ and hence are both $\tilde{s}$-SE's. Hence, a product of two USEP's is not necessarily a USEP. We have however the following.

Proposition 3.10 *For reflexive graphs we have:*

1. *A retract of a reflexive graph with the SEP has the SEP.*
2. *A retract of a reflexive graph with the USEP has the USEP.*

Proof. Let G be a retract of G' where G' has the SEP and $s \in V(G)$ a given vertex. Let $r : G' \to G$ be the retraction. Let H be a reflexive graph, $U \subseteq V(H)$ and $f : U \to V(G)$ an NE-map. Since $V(G) \subseteq V(G')$ we can view f as a map from U into $V(G')$. By the SEP of G' there is an s-SE $\phi'_{f;s} : H \to G'$. Let $\phi_{f;s} = r \circ \phi'_{f;s} : H \to G$. Clearly $\phi_{f;s}$ is a homomorphism that extends f. Assume that $\phi_f : H \to G$ is another extension of f. Since r is a retraction and hence an NE-map, we have

$$d_G(\phi_{f;s}(x), s) = d_G((r \circ \phi'_{f;s})(x), r(s)) \leq d_{G'}(\phi'_{f;s}(x), s). \tag{4}$$

Since G is an isometric subgraph of G' and by viewing ϕ_f as a homomorphism into G' we have, since $\phi'_{f;s}$ is an s-SE, that

$$d_{G'}(\phi'_{f;s}(x), s) \leq d_{G'}(\phi_f(x), s) = d_G(\phi_f(x), s). \tag{5}$$

By (4) and (5) we have that $d_G(\phi_{f;s}(x), s) \leq d_G(\phi_f(x), s)$. Since $s \in V(G)$ was arbitrary, we have that G has the SEP. Hence, we have proved the first part.

Assume further that G' has the USEP. We want to show that $\phi_{f;s} : H \to G$ is the unique s-SE of G. Let $\mathbf{i} : G \hookrightarrow G'$ be the inclusion map.

Claim 3.11 *The homomorphism $\mathbf{i} \circ \phi_{f;s} : H \to G'$ is an s-SE of G'.*

Proof. (Claim 3.11:) Since G is an isometric subgraph of G' we have

$$d_{G'}((\mathbf{i} \circ \phi_{f;s})(x), s) = d_G(\phi_{f;s}(x), s). \tag{6}$$

Assume $\theta_f : H \to G'$ extends $f : U \to V(G) \subseteq V(G')$. Since the retraction r is a homomorphism it is nonexpansive and hence

$$d_G((r \circ \theta_f)(x), s) \leq d_{G'}(\theta_f(x), s). \tag{7}$$

Clearly $r \circ \theta_f$ extends the vertex map f. Since $\phi_{f;s}$ is an s-SE of G, then we have by (6) and (7) that $d_{G'}((\mathbf{i} \circ \phi_{f;s})(x), s) = d_G(\phi_{f;s}(x), s) \leq d_G((r \circ \theta_f)(x), s) \leq d_{G'}(\theta_f(x), s)$. This completes the proof of Claim 3.11. □

By the USEP property of G' and Claim 3.11 the sink-extension $\mathbf{i} \circ \phi_{f;s}$ is unique and hence so is $\phi_{f;s}$. This completes the proof of the second part. □

Note that in a product $\widetilde{G} = G_1 \times \ldots \times G_k$ each G_i can be viewed as a retract of $\widetilde{G}$. Hence, by Proposition 3.10, we can state the reverse of Observation 3.9.

Corollary 3.12 *A product $\widetilde{G} = G_1 \times \ldots \times G_k$ of reflexive graphs has the SEP if, and only if, each G_i has the SEP.*

Let G be a reflexive graph satisfying the EP. By Observation 2.3 and Theorem 3.8, G must be a retract of product of reflexive paths. By Corollary 3.6, Observation 3.9 and Proposition 3.10, G must have the SEP. Hence, we have the following.

Theorem 3.13 *A reflexive graph has the SEP if, and only if, it has the EP.*

We summarize in the following extension of Observation 2.3.

Corollary 3.14 *For a reflexive graph G the following are equivalent:*

1. *G has the SEP.*
2. *G has the EP.*
3. *G is an absolute retract.*
4. *G has the Helly property.*

Algorithmic concerns: To decide whether a given reflexive graph on n vertices and m non-loop edges has the Helly property can be done in polynomial time. In fact, Bandelt and Pesch [2] gave two recognition algorithms for Helly graphs, one of time complexity $O(n^4)$ and the other of time complexity $O(mn^2)$. Hence, by Corollary 3.14, reflexive graphs satisfying the SEP can also be recognized by these very polynomial time algorithms.

Although not entirely within the focus of this paper, we will conclude this section by briefly considering the relevant special case when G is a reflexive path $G = P$:

If P has n vertices, then we can represent the vertices of P by the integers $\{1, \ldots, n\} = [n]$ and connect a and b iff $|a - b| \leq 1$. In this way we obtain a total order on $V(P) = [n]$. Hence, if X is a set, then we have a natural partial order $\leq'$ on the collection of all vertex maps $f : X \to P$, namely by putting $f_1 \leq' f_2$ iff $f_1(x) \leq f_2(x)$ holds in $V(P) = [n]$ for all $x \in X$. In [5, Thm. 2] it is shown that if H is a reflexive graph, $U \subseteq V(H)$ and $f : U \to P$ an NE-map, then there are homomorphisms $\phi^-, \phi^+ : H \to P$ extending f such that for *any* homomorphism ϕ that also extends f we have $\phi^- \leq' \phi \leq' \phi^+$. By using the SEP of reflexive paths and their products, this becomes clear and transparent, and can further be generalized as we will now briefly do:

Recall that for $k \in \mathbb{N}$ we have a natural partial order $\preceq$ on $\mathbb{N}^k$ given by

$$\tilde{a} \preceq \tilde{b} \Leftrightarrow a_i \leq b_i \text{ for all } i \in [k].$$

Since each reflexive path on n vertices can be represented by the vertices $[n] \subseteq \mathbb{N}$, we can embed any product of reflexive paths $P_1 \times \cdots \times P_k$ into the infinite reflexive grid $\mathbb{N}^k$ in which two vertices $\tilde{a}$ and $\tilde{b}$ are adjacent iff $\max_{i \in [k]}\{|a_i - b_i|\} \leq 1$. With this notation we have the following.

Proposition 3.15 *Let H be a (finite) connected reflexive graph, $U \subseteq V(H)$ and $f : U \to \mathbb{N}^k$ a NE-map. Then there are homomorphisms $\phi_{f;\min}, \phi_{f;\max} : H \to \mathbb{N}^k$ extending f, such that for any homomorphism ϕ_f extending f we have $\phi_{f;\min} \preceq \phi_f \preceq \phi_{f;\max}$.*

Proof. (Sketch.) Since H is finite and connected, then there is an $m \in \mathbb{N}$ such that $f : U \to [m]^k \subseteq \mathbb{N}^k$ and such that any homomorphism ϕ_f extending f has $\phi_f(x) \preceq \tilde{m} = (m, \ldots, m) \in \mathbb{N}^k$. By Theorem 3.9 we have that $[m]^k$ has the SEP, so the desired bounding homomorphisms can obtained by letting $\phi_{f;\min}$ be the $\tilde{1}$-SE $\phi_{f;\tilde{1}}$ and $\phi_{f;\max}$ the $\tilde{m}$-SE $\phi_{f;\tilde{m}}$, where $\tilde{1} = (1, \ldots, 1) \in \mathbb{N}^k$. □

When $k = 1$, Proposition 3.15 is now precisely [5, Thm. 2]. In fact, for $k = 1$ we can further say the following. The proof is clear by the definition of an s-SE given in the proof of Theorem 3.5.

Proposition 3.16 *Let H be a (finite) connected reflexive graph, $U \subseteq V(H)$ and $f : U \to \mathbb{N}$ a NE-map. In this case all the i-SE's $\{\phi_{f;i} : i \in \mathbb{N}\}$ are totally ordered and form a chain*

$$\phi_{f;1} \leq' \phi_{f;2} \leq' \cdots \leq' \phi_{f;i} \leq' \cdots.$$

Moreover, this listing eventually becomes stationary, that is, for each H there is an $N \in \mathbb{N}$ that depends on H, U and f, such that $\phi_{f;i} = \phi_{f;N}$ for all $i \geq N$.

4 On graphs satisfying the USEP

As we saw in the example preceding Proposition 3.10, the product $P_1 \times P_2$ of two reflexive paths on two and three vertices respectively did not satisfy the USEP. By the same token as in that example, we can argue that any reflexive graph that contains the reflexive 4-cycle C_4 with at most one chord as an induced subgraph does not have the USEP. Therefore, if G_1 and G_2 are two connected reflexive graphs on two or more vertices and G_2 is not complete, then $G_1 \times G_2$ does not have the USEP. Since the class of connected reflexive graphs has the unique factorization property w.r.t the product $\times$, then we have in particular the following observation. (Note that by a *prime* graph we mean a connected reflexive graph that cannot be written as a product of two graphs, each on two or more vertices and neither complete. For more detailed information see [14, p. 159] or the original paper [17].)

Observation 4.1 *A connected reflexive graph satisfying the USEP is either a prime graph or a complete reflexive graph.*

The only fact we used to deduce Observation 4.1 was that if there are two vertices with at least two shortest paths between them, then the graph cannot satisfy the USEP. This we state more formally.

Lemma 4.2 *If G is a connected reflexive graph that satisfies the USEP, then the shortest path between any pair of vertices in G is unique.*

Proof. (Sketch.) Assume there are two vertices $x, y \in V(G)$ with two different shortest paths P and P' between them of length $k = d_G(x, y)$. Let H be the simple reflexive path on $u_0, u_1, \ldots, u_k$, $U = \{u_0, u_k\} \subseteq V(H)$ and $f : U \to V(G)$ be given by $f(u_0) = x$ and $f(u_k) = y$. In this case mapping H onto either P or P' will in both cases yield an s-SE for $s \in \{x, y\}$. Hence, there are at least two s-SE's in G. □

The converse of Lemma 4.2 does not hold since the shortest path between any pair of distinct vertices in an odd cycle C_m is unique, and it is easy to see that C_m does not have the EP for any $m \geq 4$.

For a more complete description of graphs satisfying the USEP we need the following intuitively obvious theorem.

Theorem 4.3 *If G is a reflexive Helly graph in which the shortest path between any pair of vertices is unique, then each cycle of G induces a clique in G.*

Proof. Let G be a reflexive Helly graph in which shortest paths are unique. Let C_m be a cycle of length $m \in \mathbb{N}$. We will show by induction on m that C_m must induce a clique:

For $m = \{1, 2, 3\}$ then C_m is a reflexive clique, so we may assume $m \geq 4$. If $m = 4$ and $C_m = C_4$ does not induce a clique, then there are two opposite vertices of C_4 that are not connected with a cord in G. In this case there are at least two shortest 2-paths between them, a contradiction. Hence we can assume $m \geq 5$. We start by showing that C_m must have a cord. If

$$\begin{array}{ll} x_1 = u_0 & \ell_1 = \lfloor (m-3)/2 \rfloor \\ x_2 = u_{\lfloor (m-1)/2 \rfloor} & \ell_2 = 1 \\ x_3 = u_{\lceil (m+1)/2 \rceil} & \ell_3 = 1 \end{array}$$

then the closed balls $N^G_{\ell_1}[x_1]$, $N^G_{\ell_2}[x_2]$ and $N^G_{\ell_3}[x_3]$ have pairwise a nonempty intersection, so by the Helly property of G they have a nonempty intersection. Let v be a vertex in their intersection. Since either $v \neq x_2$ or $v \neq x_3$, we can for symmetric reasons assume that $v \neq x_2$. Also, we may assume that $v \notin \{u_0, u_1, \ldots, u_{\lfloor (m-3)/2 \rfloor}\}$ since otherwise $\{x_3, v\}$ is a cord of C_m. Hence, we can assume that $v \notin \{u_0, u_1, \ldots, u_{\lfloor (m-1)/2 \rfloor}\}$.

Claim 4.4 *Let $p \geq 2$, $q \leq p$ and $P = (u_0, u_1, \ldots, u_p)$ and $Q = (u_0, v_1, \ldots, v_q)$ be paths in G satisfying $u_p = v_q$, $u_{p-1} \neq v_{q-1}$ and $d_G(u_0, v_{q-1}) = q - 1$. In this case the circuit PQ^{-1} contains a cycle involving at least three vertices from P.*

Proof. We may assume that $q \geq 2$ since otherwise P induces a cycle and we are done. Otherwise, the claim is clearly true for $p = 2$ since in that case PQ^{-1} is itself a cycle. For $p \geq 3$ we proceed by induction as follows:

If $u_1 \in \{v_1, \ldots, v_{q-1}\}$, then $u_1 \notin \{v_2, \ldots, v_{q-1}\}$ since otherwise we have $d_G(u_0, v_{q-1}) < q - 1$. Hence $u_1 = v_1$ must hold and the claim follows by induction on $P' = (u_1, \ldots u_p)$ and $Q' = \{v_1, \ldots, v_q\}$.

Otherwise there is a least index $\alpha \in \{2, \ldots, p\}$ such that $u_\alpha \in \{v_2, \ldots, v_q\}$, say $u_\alpha = v_\beta$. (Note that $\beta \leq \alpha$.) If $P_\alpha = (u_0, u_1, \ldots, u_\alpha)$ and $Q_\beta = (u_0, v_1, \ldots, v_\beta)$ then $P_\alpha Q_\beta^{-1}$ is a cycle involving u_0, u_1 and u_2. This completes the proof of the claim. ◻

Continuing with our proof of Theorem 4.3, let $P = (u_0, u_1, \ldots, u_{\lfloor (m-1)/2 \rfloor})$ be a path from u_0 to $u_{\lfloor (m-1)/2 \rfloor} = x_2$ and let $Q = (u_0, v_1, \ldots, v_q)$ be a path from u_0 to $v_q = x_2$ where $v_{q-1} = v$ and $d_G(u_0, v_{q-1}) = q - 1$. By Claim 4.4 PQ^{-1} contains a cycle involving three of the vertices from P. This cycle has at most $2\lfloor (m-1)/2 \rfloor \leq m - 1$ vertices and hence by induction hypothesis this cycle must induce a clique in G. Further, this cycle involves at least three vertices of C_m which must therefore have a chord, say $\{u_0, u_k\}$ where $k \in \{2, \ldots, m-2\}$ and where $C_m = (u_0, \ldots, u_{m-1}, u_0)$. By induction hypothesis the cycles $C' = (u_0, \ldots, u_k, u_0)$ and $C'' = (u_0, u_k, \ldots, u_{m-1}, u_0)$ each induces a clique in G. Therefore, if $i \in \{1, \ldots, k-1\}$ and $j \in \{k+1, \ldots, m-1\}$ then the four-cycle $C''' = (u_0, u_i, u_k, u_j, u_0)$ also induces a clique in G, so u_i and u_j are connected in G. This shows that C_m induces a clique in G and completes the induction and the proof. ◻

A graph in which every cycle induces a clique has all its blocks as cliques. By Theorem 4.3 we therefore have the following:

Corollary 4.5 *A reflexive Helly graph in which the shortest path between any pair of vertices is unique must be a block-tree.*

By Theorem 3.5, Lemma 4.2 and Corollaries 3.14 and 4.5 we therefore have the following complete characterization.

Theorem 4.6 *A connected reflexive graph G has the USEP if, and only if, G is a block-tree.*

Algorithmic concerns: Recognition of block-trees is easy: A pair of distinct vertices are in the same block iff they are adjacent. Hence, by Theorem 4.6, graphs with the USEP can be recognized in $O(n^2)$-time.

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