Helly-Type Theorems on Support Lines for Families of Congruent Disks in the Plane

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Overview

1. Motivation
   - Helly's theorem and Helly-type theorems
   - Helly-type problem for support lines
   - Specializing to unit disks

2. Description of Helly-type problem, main results and methods
   - Two Helly-type theorems
   - Constructing critical families $\mathcal{F}_4$
   - Nonextendable critical families and proof methods

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Primary motivation: Helly's theorem

Theorem (Helly's Theorem)

Suppose $\mathcal{F}$ is a family of at least $n+1$ convex sets in the n-dimensional vector space $\mathbb{R}^n$ such that $\mathcal{F}$ is finite or each member of $\mathcal{F}$ is compact. If each $n+1$ members of $\mathcal{F}$ have a common point, there is a point common to all members of $\mathcal{F}$.

Primary motivation: Helly-type theorems

For Helly-type results, a (uniform) local property implies a global property.

If every subfamily of size k of the family $\mathcal{F}$ has property P, then the family has the property P.

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The Helly-type problem for support lines of convex bodies

A special class of transversals, namely *support lines*, was introduced by Dawson [1].

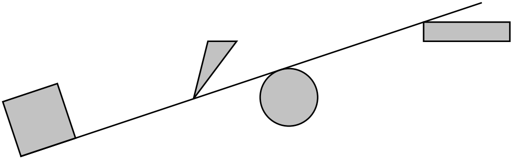

Figure: Support line for a family of convex bodies in the plane.

We can view a *support line* as an extremal version of a transverse line.

The support property, notation and related concepts

A family $\mathcal{K}$ of convex sets or bodies in the plane has the (support) property S provided there is a line supporting every member of $\mathcal{K}$.

Similarly, $\mathcal{K}$ has the (support) property $S(k)$ provided every subfamily of k members from $\mathcal{K}$ admits a common support line.

Helly-type problem for support lines

For a family $\mathcal{K}$ of convex sets in the plane, when does the implication $S(k) \implies S$ hold?

Partial summary of Helly-type results for support lines to disjoint convex bodies

Dawson [1] proved the assertions below. Revenko and Soltan [3] provide the improvement of the last assertion in the following theorem.

Theorem

For a finite disjoint family $\mathcal{K}$ of convex bodies in the plane, one has
$S(5) \implies S, \quad S(4) \implies S \text{ if } |\mathcal{K}| \geq 7, \quad S(3) \implies S \text{ if } |\mathcal{K}| \geq 143.$

Open Helly-type problem

The following problem, formulated in Revenko and Soltan [3], is still open.

Problem

Find the smallest value of the natural number n such that $S(3) \implies S$ for any disjoint family of n or more convex bodies in the plane.

Proof of concept: hard problem (from Revenko and Soltan [3])

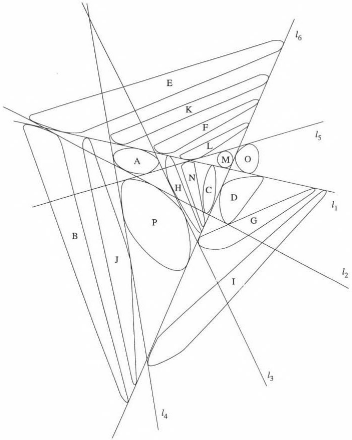

Figure: Sixteen convex bodies with the property $S(3)$ but not S.

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The Helly-type problem for support lines to disjoint unit disks

For a family $\mathcal{F}$ of disjoint unit disks in the plane, when does the implication $S(k) \implies S$ hold?

The results of V. Soltan [4] are summarized together in the following slide.

Motivation to specify to unit disks:

We can interpret unit disks as an ε-ball around a convex body.

As seen, the general problem for convex bodies is hard.

Six disjoint disks with $S(3)$ but not S

The Figure below (V. Soltan, 2023) shows that the property $S(3)$ does not imply S for any disjoint family $\mathcal{F}$ with 6 or fewer congruent disks.

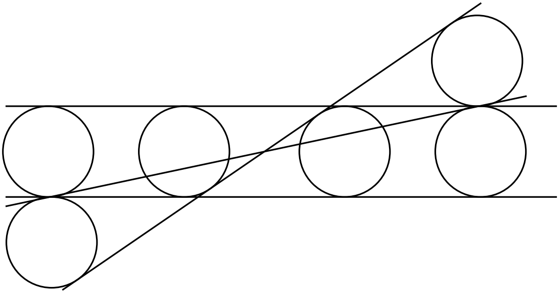

Figure: $S(3) \not\Rightarrow S$ if $|\mathcal{F}| \leq 6$.

Summary of the Helly-type results in V. Soltan [4]

Theorem

If $\mathcal{F}$ is a disjoint family (possibly infinite) of unit disks in the plane, then $S(4) \implies S$. Furthermore, $S(3) \implies S$ provided that $|\mathcal{F}| \geq 7$.

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Generalizing to nonoverlapping congruent disks

A generalization of the problem studied in V. Soltan [4].

Problem

Given a nonoverlapping family of congruent disks in the plane, show that $S(4) \implies S$*. Find the smallest value of the natural number n such that* $S(3) \implies S$ *for any nonoverlapping family of n or more congruent disks in the plane.*

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First Helly-type theorem on support lines

Theorem

For any nonoverlapping family $\mathcal{F}$ of congruent disks in the plane, one has $S(4) \implies S$.

Second Helly-type theorem on support lines

Theorem

For a nonoverlapping family $\mathcal{F}$ of congruent disks in the plane, $S(3) \implies S$ if the family has 8 or more members.

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Definition of critical family

Definition

A nonoverlapping (touching, disjoint) family $\mathcal{F}$ of congruent disks in the plane with the property $S(3)$ but not $S(4)$ is called *critical*.

Reduction to critical families $\mathcal{F}_4$

Any critical nonoverlapping family $\mathcal{F}_n$ ($n \geq 5$) of congruent disks in the plane contains a critical nonoverlapping subfamily $\mathcal{F}_4$.

This holds since $\mathcal{F}_n$ has $S(3)$ and not $S(4)$, so that some four disks are not supported by a line and this subfamily retains property $S(3)$.

Any $\mathcal{F}_3$ with $S(3)$ has S, not interesting.

A minimal critical family has size 4.

Family $\mathcal{L}_{ij}$ of support lines for disks C_i, C_j

The following figure gives a useful example of the notation $\mathcal{L}_{ij}$.

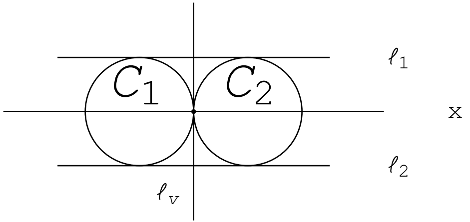


Figure: Touching family $\{C_1, C_2\}$ and its supports $\mathcal{L}_{12} = \{\ell_1, \ell_2, \ell_v\}$.

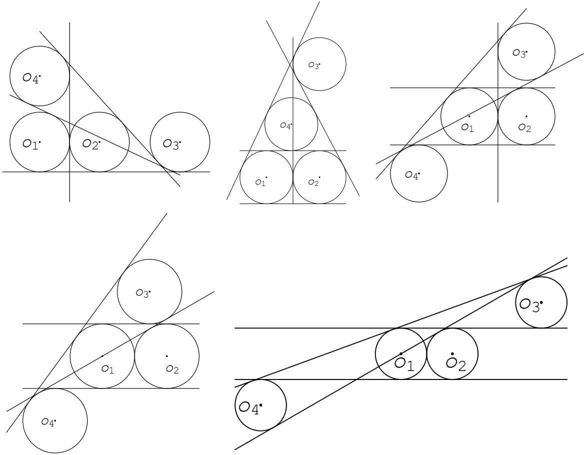
Outline of constructing critical families of size 4

- Begin with $\{C_1, C_2\}$ (which has symmetries of the rectangle, Klein four-group V).

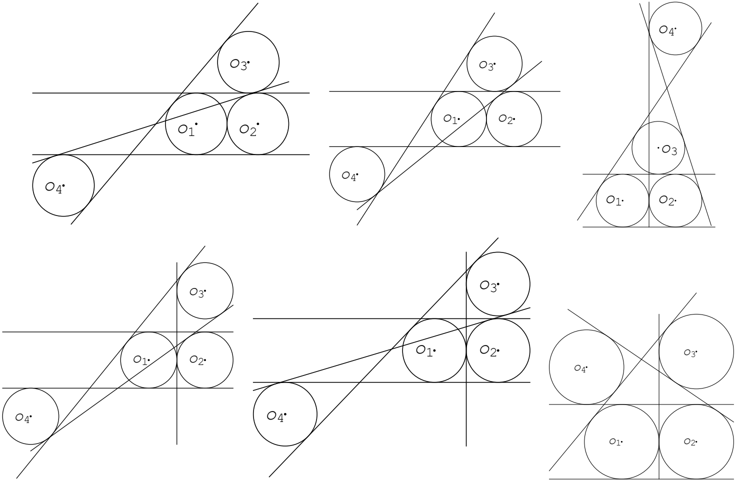
- At least one line in $\mathcal{L}_{12} = \{\ell_1, \ell_2, \ell_v\}$ must support each of C_3, C_4.

- Place two additional disks in the plane and solve a resulting system of equations to determine the coordinates of the centers $o_3(\gamma, y_3)$ and $o_4(x_4, y_4)$ of the disks C_3 and C_4 (and verify the coordinates are real).

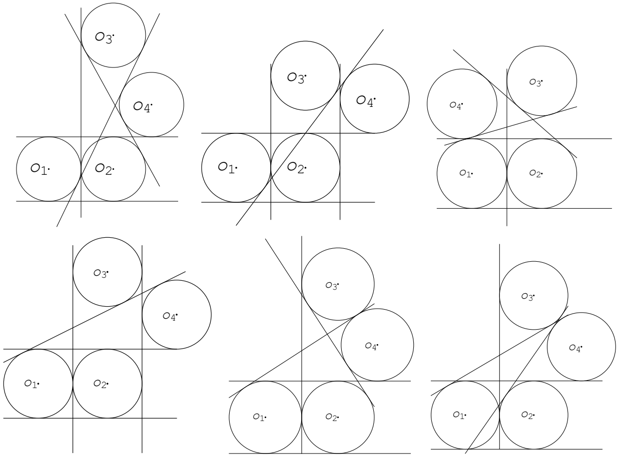
Critical $\mathcal{F}_4$ with three disks in a slab



Six critical families $\mathcal{F}_4$ avoiding three disks in a slab



Remaining 6 critical families $\mathcal{F}_4$ avoiding three disks in a slab



All 17 critical families $\mathcal{F}_4$ shown in the preceding slides are pairwise distinct

Outline of proof idea:

- Use the group of symmetries of the disks $\{C_1, C_2\}$ and lines $\mathcal{L}_{12}$: the Klein four-group V.
- Verify the families are distinct up to rotations and reflections in V.

Reduction to critical families $\mathcal{F}_4$ (Repeated Slide)

Any critical nonoverlapping (disjoint) family $\mathcal{F}_n$ ($n \geq 5$) of congruent disks in the plane contains a critical nonoverlapping (disjoint) subfamily $\mathcal{F}_4$.

Since $\mathcal{F}_n$ has $S(3)$ and not $S(4)$, some four disks are not supported by a line but retain property $S(3)$.

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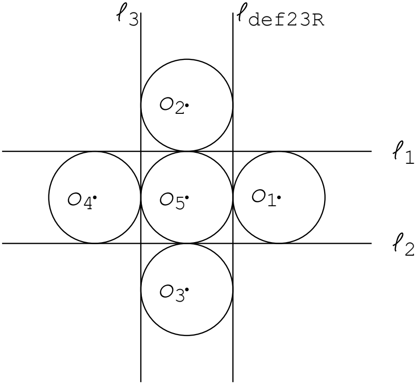
Unique touching critical family $\mathcal{F}_5$ with no touching critical subfamily $\mathcal{F}_4$ (See Addendum II for a constructive proof.)



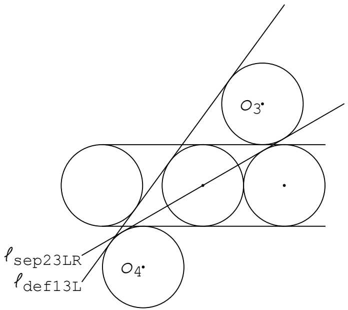
Extending touching critical families $\mathcal{F}_4$

Heuristically, seeking extensions $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C_5\}$ directly, we must examine at least 955 configurations of support lines.

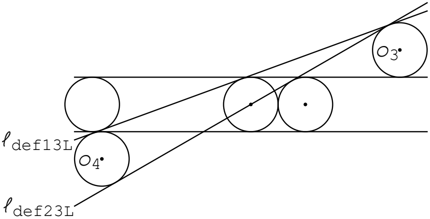
We avoid this by first showing that precisely 2 critical supports of $\mathcal{F}_4$ must support C_5.

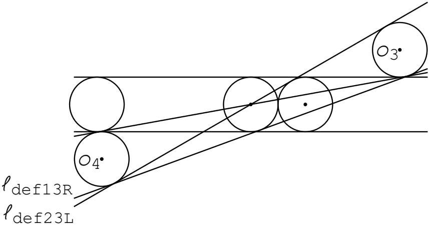
To show this, observe that if a single critical support of $\mathcal{F}_4$ supports C_5, then $\mathcal{F}_5$ necessarily has at least 2 critical subfamilies of size 4. In the paper, we show this is impossible.

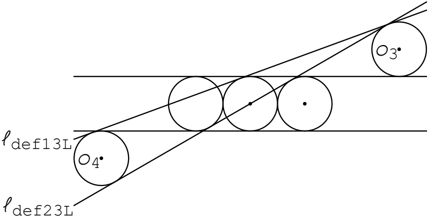
Extensions of size 5 of the family depicted in Figure 3.21d



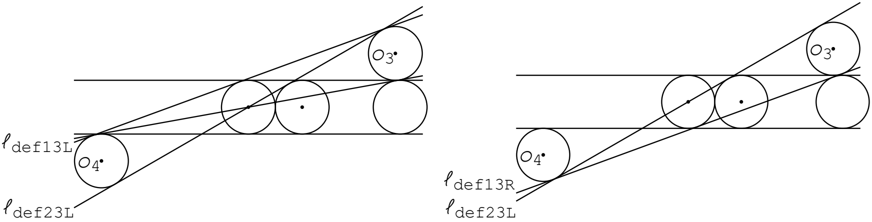
Three of five extensions of size 5 of the family depicted in Figure 3.21e



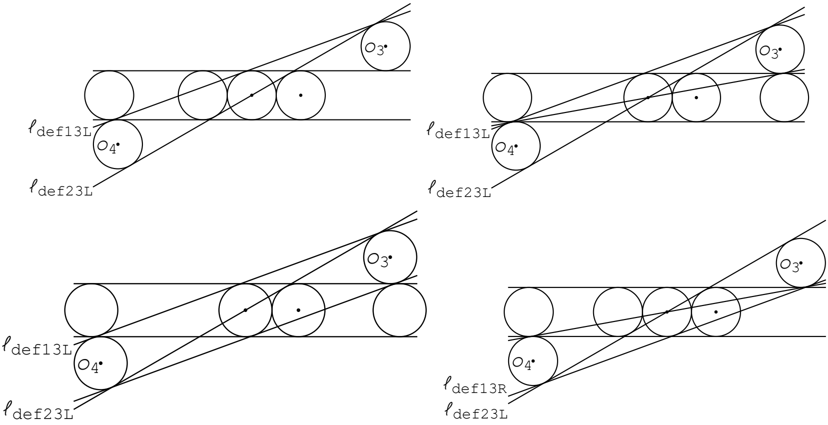




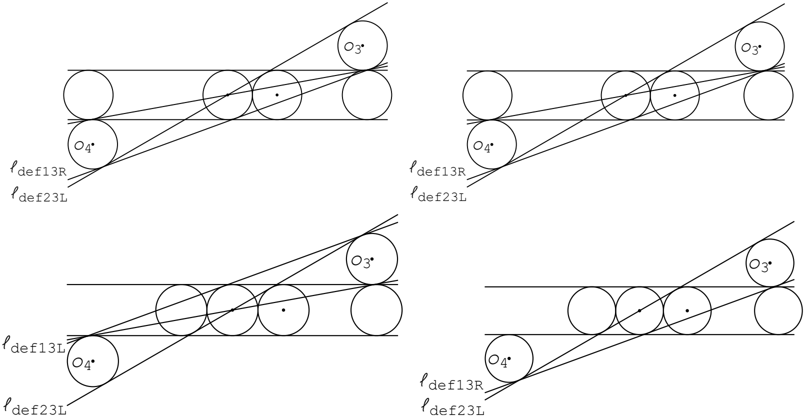
Two of five extensions of size 5 of the family depicted in Figure 3.21e



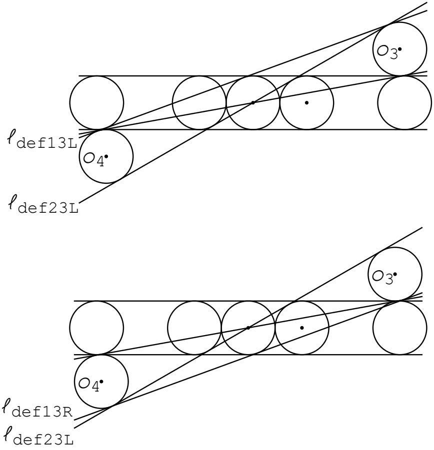
Four of eight extensions of size 6 of the family depicted in Figure 3.21e



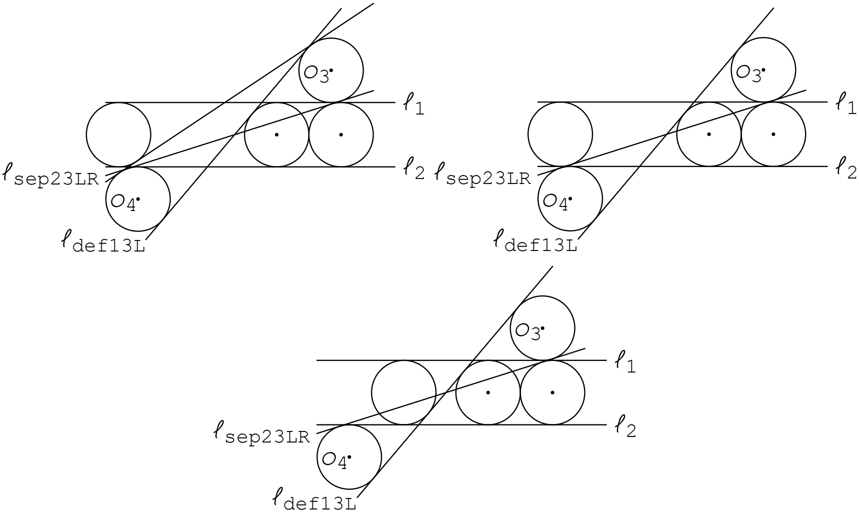
Remaining four of eight extensions of size 6 of the family depicted in Figure 3.21e



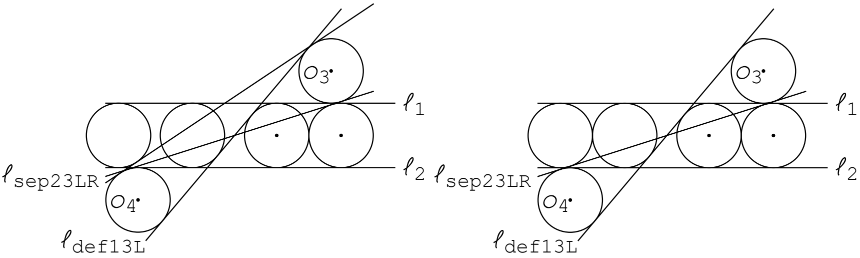
Extensions of size 7 of the family depicted in Figure 3.21e



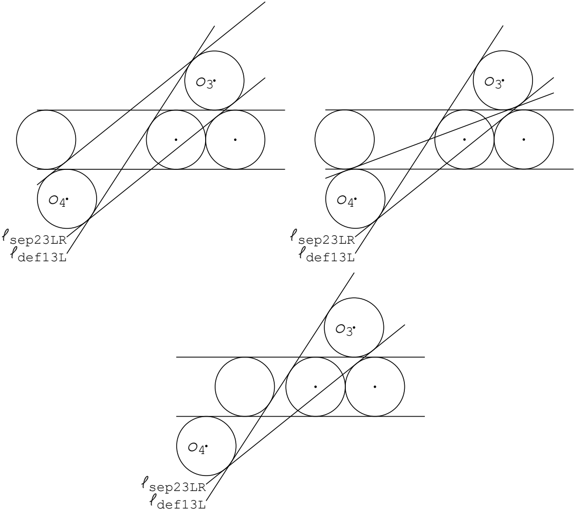
Extensions of size 5 of the family depicted in Figure 3.22a



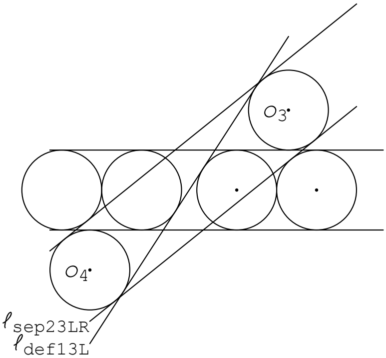
Extensions of size 6 of the family depicted in Figure 3.22a



Extensions of size 5 of the family depicted in Figure 3.22b



Extension of size 6 of the family depicted in Figure 3.22b



Outline of the proof of the second Helly-type theorem

The theorem states $S(3) \implies S$ if the nonoverlapping family $\mathcal{F}$ contains at least 8 members.

- Survey all of the critical nonoverlapping families.
- Identify the largest family (size 7).
- It follows that any nonoverlapping family $\mathcal{F}$ of at least 8 members with property $S(3)$ has the support property S (by exhaustion).

References

R. Dawson, Helly-type theorems for bodies in the plane with common supports, Geom. Dedicata **45** (1993), 289–299.

E. Helly, Über Mengen konvexer Körper mit gemeinschaftlichen Punkten, Jber. Deutsch. Math. Verein. **32** (1923), 175–176.

S. Revenko, V. Soltan, Helly-type theorems on transversality for set-system, Studia Sci. Math. Hungar. **32** (1996), 395–406.

V. Soltan, Helly-type results on support lines for disjoint families of unit disks, Beiträge Algebra Geom. **61** (2020), 139-150.

V. Soltan, Support lines for disjoint families of unit disks, Beiträge Algebra Geom. **64** (2023), 531–534.

Thank you.

Addendum I: Connection to topology

Helly's theorem, and Helly-type theorems, of combinatorial geometry share an affinity with the finite intersection property (f.i.p.) of standard topology.

Definition (*Topology*, James R. Munkres, p. 169)

A collection $\mathcal{C}$ of subsets of [a space] X is said to have the **finite intersection property** if for every finite subcollection $\{C_1, \ldots, C_n\}$ of $\mathcal{C}$, the intersection $C_1 \cap \cdots \cap C_n$ is nonempty.

Addendum II: Extension $\mathcal{F}_n$ from disjoint critical $\mathcal{F}_4$

Assume first that $n = 5$.

Suppose the extension $\mathcal{F}_5 = \mathcal{F}_4 \cup \{C_5\}$ contains disjoint critical family $\mathcal{F}_4 = \{C_1, C_2, C_3, C_4\}$.

And, a line supports each of its touching subfamilies of size four.

Addendum II: Touching subfamilies of size 4

Since C_5 touches at least one disk of $\mathcal{F}_4$,
we stipulate up to labels that C_5 touches C_4.

The touching subfamilies of size four contain $\{C_4, C_5\}$ and have form $\{C_4, C_5\} \cup \{C_i, C_j\}$ $(i \neq j \in \{1, 2, 3\})$.

Precisely $1 \cdot \binom{3}{2} = 3$ touching subfamilies $\mathcal{F}, \mathcal{G}, \mathcal{H}$ of size four in $\mathcal{F}_5$ contain subfamily $\{C_4, C_5\}$.

Addendum II: Touching subfamilies $\mathcal{F}, \mathcal{G}, \mathcal{H}$

For notational convenience, we fix the touching subfamilies by label as in the following:

$$\mathcal{F} = \{C_4, C_5\} \cup \{C_1, C_2\} = \{C_1, C_2, C_4, C_5\}$$
$$\mathcal{G} = \{C_4, C_5\} \cup \{C_1, C_3\} = \{C_1, C_3, C_4, C_5\}$$
$$\mathcal{H} = \{C_4, C_5\} \cup \{C_2, C_3\} = \{C_2, C_3, C_4, C_5\}$$

Addendum II: Touching subfamily $\{C_4, C_5\}$ of $\mathcal{F}_5$

Touching $\{C_4, C_5\}$ has three support lines $\mathcal{L}_{45} = \{\ell_1, \ell_2, \ell_3\}$.

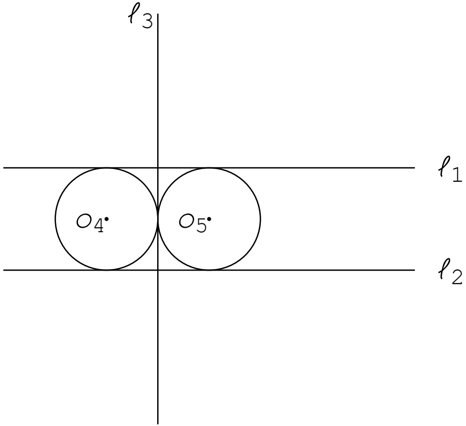


Figure: Touching subfamily $\{C_4, C_5\}$ of the extension $\mathcal{F}_5$ of disjoint critical $\mathcal{F}_4$.

Addendum II: Support lines in $\mathcal{L}_{45}$

A line supports each subfamily $\mathcal{F}, \mathcal{G}, \mathcal{H}$, and these lines are necessarily in $\mathcal{L}_{45}$ since $\{C_4, C_5\}$ belongs to each of $\mathcal{F}, \mathcal{G}, \mathcal{H}$.

Each of the three lines in $\mathcal{L}_{45}$ supports precisely one of the subfamilies $\mathcal{F}, \mathcal{G}, \mathcal{H}$ (by the pigeonhole principle).

Addendum II: Support relations

Up to labels, we stipulate the following:

$$\begin{aligned} \ell_1 \text{ supports } \mathcal{F} &\implies \ell_1 \text{ supports } \{C_1, C_2, C_4\} \\ \ell_2 \text{ supports } \mathcal{G} &\implies \ell_2 \text{ supports } \{C_1, C_3, C_4\} \\ \ell_3 \text{ supports } \mathcal{H} &\implies \ell_3 \text{ supports } \{C_2, C_3, C_4\} \end{aligned}$$

The preceding implies the following pairs of lines support each respective disk as labeled:

$$\begin{aligned} &\text{Both of } \ell_1, \ell_2 \text{ support } C_1, \\ &\text{both of } \ell_1, \ell_3 \text{ support } C_2, \qquad \text{and} \\ &\text{both of } \ell_2, \ell_3 \text{ support } C_3. \end{aligned}$$

Addendum II: Three of 4 disks in disjoint critical $\mathcal{F}_4$

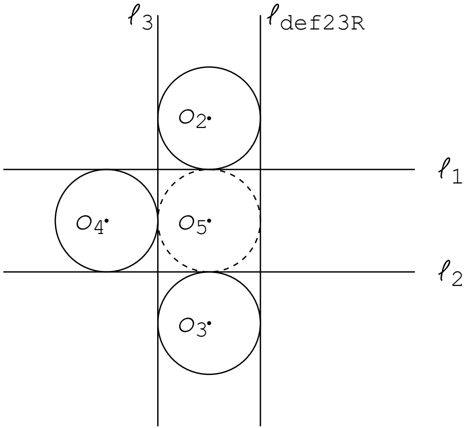


Figure: Induced positions of C_2, C_3 disjoint from C_4.

Addendum II: The family $\mathcal{F}_5$

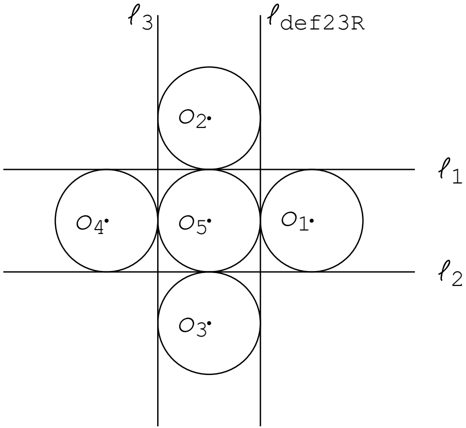
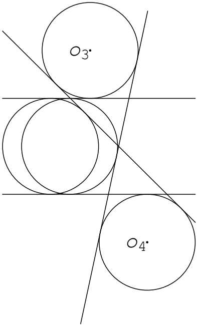
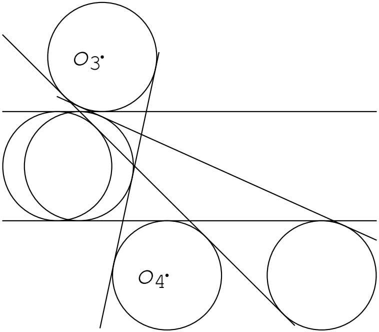


Figure: Unique touching critical $\mathcal{F}_5$ with no touching critical $\mathcal{F}_4$.

Addendum III: Overlapping family $\mathcal{F}_4$ that is extendable with C_5 not in the slab between the parallel critical supports



Addendum III: Overlapping extension of size 5



Addendum IV: Conjecture: $S(3) \implies S$ has threshold number 10 for overlapping families

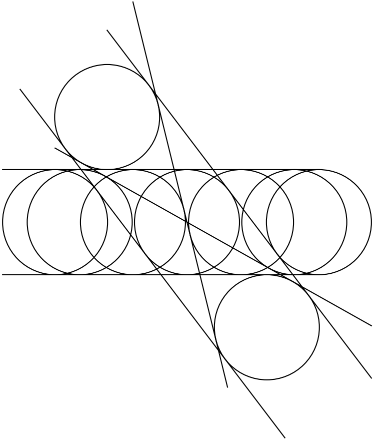


Figure: Overlapping critical $\mathcal{F}_9$.

End of slides.