When does the content map on polynomials act like a homomorphism, and how broadly does the answer apply?

Neil Epstein, George Mason University, Fairfax VA – 22030

Abstract

Let $R$ be a commutative ring. When $S = R[x]$, one defines the content $c(f)$ of an element $f \in S$ as the ideal of $R$ generated by its coefficients, and $f$ is Gaussian if $c(fg) = c(f)c(g)$ for all $g \in S$. Lucas answered two longstanding questions when he established in 2005 and 2008 respectively that

(a) if $f \in S$ is not a zero divisor, then it is Gaussian if and only if the content ideal $c(f) \subseteq R$ is locally principal, and that

(b) if $R$ is reduced, then for any $f \in S$, $f$ is Gaussian if and only if $c(f)$ is locally principal.

This generalized earlier work of Loper, Roitman, Gilmer, Heinzer, Huneke, and others, tracing back ultimately to Gauss. However, the notion of content also makes sense when $S$ is a polynomial extension of $R$ in several variables, or more generally an affine semigroup algebra over $R$, or a power series extension over $R$. If $R$ is a $K$-algebra where $K$ is a field, and $L/K$ is a well-behaved field extension, one may even make sense of content of elements of $S = R \otimes_K L$ over $R$, with respect to a vector-space basis of $L$ over $K$. In all of these contexts, and more generally, we show results analogous to (a) and (b) above, provided certain assumptions about Noetherianess, the property of being “approximately Gorenstein”, and module-freeness. This is done by use of the Ohm-Rush content function (which subsumes all the notions of content mentioned above) and the coordinated notion of a semicontent algebra.

This work is joint with Jay Shapiro.

Keywords: content, Gaussian polynomial, polynomial extension, power series extension, Ohm-Rush content function.