When does the content map on polynomials act like a homomorphism, and how broadly does the answer apply?

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Abstract

Let R be a commutative ring. When $S = R[x]$, one defines the *content* $c(f)$ of an element $f \in S$ as the ideal of R generated by its coefficients, and f is *Gaussian* if $c(fg) = c(f)c(g)$ for all $g \in S$. Lucas answered two longstanding questions when he established in 2005 and 2008 respectively that

(a) if $f \in S$ is not a zero divisor, then it is Gaussian if and only if the content ideal $c(f) \subseteq R$ is locally principal, and that

(b) if R is reduced, then for any $f \in S$, f is Gaussian if and only if $c(f)$ is locally principal.

This generalized earlier work of Loper, Roitman, Gilmer, Heinzer, Huneke, and others, tracing back ultimately to Gauss.

However, the notion of content also makes sense when S is a polynomial extension of R in several variables, or more generally an affine semigroup algebra over R, or a power series extension over R. If R is a K-algebra where K is a field, and L/K is a well-behaved field extension, one may even make sense of content of elements of $S = R \otimes_K L$ over R, with respect to a vector-space basis of L over K. In all of these contexts, and more generally, we show results analogous to (a) and (b) above, provided certain assumptions about Noetherianness, the property of being "approximately Gorenstein", and module-freeness. This is done by use of the Ohm-Rush content function (which subsumes all the notions of content mentioned above) and the coordinated notion of a semicontent algebra.

This work is joint with Jay Shapiro.

Keywords: content, Gaussian polynomial, polynomial extension, power series extension, Ohm-Rush content function.