How well-behaved is your ring map? The Ohm-Rush content function reconsidered

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Abstract

In the 1970s, Ohm and Rush came up with an axiomatic theory to determine how close a faithfully flat ring map $R \to S$ is to acting like the polynomial extension map $R \to R[x]$. The key idea is to generalize the notion of the "content" of a polynomial (i.e. the ideal generated by its coefficients) to an element of S and then see which formulas the function satisfies. In increasing order of specificity, one may ask (with some updated terminology) whether a faithfully flat R-algebra S is (1) Ohm-Rush, (2) weak content, (3) semicontent, (4) content, or (5) Gaussian. Dedekind and Mertens (1892) showed that the polynomial extension is a content algebra. Prüfer showed it is Gaussian whenever R is a Dedekind domain (and sort of conversely). In earlier work, we showed that the power series extension map $R \to R[[x]]$ is a content algebra whenever R is Noetherian, thereby correcting a 36-year error in the literature. – The question arises then of when some or all of these conditions are equivalent. We still have no examples to show that, among faithfully flat ring maps, any of the middle three properties are distinct from each other. However, we show that if R is Noetherian and $R \to S$ is weak content, it is semicontent. Over an Artinian ring, conditions 2-4 are equivalent. In the case where R is a Dedekind domain, a field, or a valuation ring, even conditions 2-5 are equivalent. As is often the case, more structural information is available in the case where the base ring is (or both rings are) Prüfer or Dedekind. We get a particularly nice description of content in the case of extensions of the form $K[x] \to L[x]$.

Keywords: faithfully flat ring, content algebra, Noetherian ring, Artinian ring, Dedekind domain, Prüfer domain.