Generalized depth in the perfect closure

George Whelan, George Mason University, Fairfax VA – 22030

Abstract

Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$. If $I \subset R$ is an ideal, $I^{[p^e]} \subset I$ is the sub-ideal generated by the p^e-th powers of elements of I, referred to as the p^e-th Frobenius power of I for $e \in \mathbb{N}$. Now let M be any R-module. The depth of M is a fundamental measure in commutative algebra which has three equivalent definitions if R is a Noetherian ring. If R is not Noetherian, however, these three definitions can all yield distinct values.

We begin with a local Noetherian ring $(R, \mathfrak{m})$ of characteristic $p > 0$, and with a motivating question: does the depth of the module $R/I^{[p^e]}$ decrease for an ideal I as e grows? We show that the answer is yes if we assume a condition on R called F-purity. This fact leads to a new notion of depth, the *stabilizing depth* of R/I.

Next we extend R to its perfect closure R^∞, which will no longer be Noetherian in general. We then consider the R^∞-module R^∞/IR^∞, and the distinct notions of non-Noetherian depth on this module over R^∞ (Here IR^∞ is the ideal in R^∞ with the same generators as I). We investigate how these generalized depth values compare to our new stabilized version on R/I over R, where we find some surprising equivalences.

Keywords: ring, local ring, Noetherian ring, Frobenius power, perfect closure.