Generalized depth in the perfect closure

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Abstract

Let \((R, \mathfrak{m})\) be a local ring of characteristic \(p > 0\). If \(I \subset R\) is an ideal, \(I^{[p^e]} \subset I\) is the sub-ideal generated by the \(p^e\)-th powers of elements of \(I\), referred to as the \(p^e\)-th Frobenius power of \(I\) for \(e \in \mathbb{N}\). Now let \(M\) be any \(R\)-module. The depth of \(M\) is a fundamental measure in commutative algebra which has three equivalent definitions if \(R\) is a Noetherian ring. If \(R\) is not Noetherian, however, these three definitions can all yield distinct values.

We begin with a local Noetherian ring \((R, \mathfrak{m})\) of characteristic \(p > 0\), and with a motivating question: does the depth of the module \(R/I^{[p^e]}\) decrease for an ideal \(I\) as \(e\) grows? We show that the answer is yes if we assume a condition on \(R\) called \(F\)-purity. This fact leads to a new notion of depth, the stabilizing depth of \(R/I\).

Next we extend \(R\) to its perfect closure \(R^\infty\), which will no longer be Noetherian in general. We then consider the \(R^\infty\)-module \(R^\infty/IR^\infty\), and the distinct notions of non-Noetherian depth on this module over \(R^\infty\) (Here \(IR^\infty\) is the ideal in \(R^\infty\) with the same generators as \(I\)). We investigate how these generalized depth values compare to our new stabilized version on \(R/I\) over \(R\), where we find some surprising equivalences.

Keywords: ring, local ring, Noetherian ring, Frobenius power, perfect closure.