Matrix rings and representations

Samuel Mendelson, George Mason University, Fairfax VA – 22030

Abstract

A classic result in non-commutative ring theory states that a ring $R$ is an $n \times n$ matrix ring if, and only if, $R$ contains $n^2$ matrix units $\{e_{ij}\}_{1 \leq i,j \leq n}$, in which case $R \cong M_n(S)$ where $S$ is a subring of $R$ that can be described completely in terms of the matrix units. A lesser known result states that a ring $R$ is an $(m+n) \times (m+n)$ matrix ring, so $R \cong M_{m+n}(S)$ for some ring $S$, if, and only if, $R$ contains three elements $a$, $b$, and $f$ satisfying the two relations $af^m + f^n b = 1$ and $f^{m+n} = 0$. In this talk, we investigate algebras over a commutative ring (or field) with elements $c$ and $f$ satisfying the two relations $c^i f^m + f^n c^j = 1$ and $f^{m+n} = 0$. Questions about the structure of $R$ and $S$ are surprisingly difficult to answer for most cases of the integers $i$, $j$, $m$, and $n$. We will consider the case when $m = n = 1$ and the gcd$(i, j) = 1$ and describe both $R$ and the underlying ring $S$. We will then generalize this description and discuss the structure of $R$ when $m, n \neq 1$ and gcd$(i, j) > 1$. Finally, we will describe for which integers, $i, j$, $R$ can be mapped to a matrix ring over its scalar field when $m = n = 1$.

Keywords: non-commutative ring, matrix units, matrix characterization.