Matrix rings and representations

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Abstract

A classic result in non-commutative ring theory states that a ring R is an $n \times n$ matrix ring if, and only if, R contains n^2 matrix units $\{e_{ij}\}_{1 \leq i,j \leq n}$, in which case $R \cong M_n(S)$ where S is a subring of R that can be described completely in terms of the matrix units. A lesser known result states that a ring R is an $(m+n) \times (m+n)$ matrix ring,so $R \cong M_{m+n}(S)$ for some ring S, if, and only if, R contains three elements a, b, and f satisfying the two relations $af^m + f^n b = 1$ and $f^{m+n} = 0$. In this talk, we investigate algebras over a commutative ring (or field) with elements c and f satisfying the two relations $c^i f^m + f^n c^j = 1$ and $f^{m+n} = 0$. Questions about the structure of R and S are surprisingly difficult to answer for most cases of the integers i, j,m, and n. We will consider the case when $m = n = 1$ and the $\gcd(i, j) = 1$ and describe both R and the underlying ring S. We will then generalize this description and discuss the structure of R when $m, n \neq 1$ and $\gcd(i, j) > 1$. Finally, we will describe for which integers, i, j, R can be mapped to a matrix ring over its scalar field when $m = n = 1$.

Keywords: non-commutative ring, matrix units, matrix characterization.