Puzzles and (many flavors of) Schubert calculus

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Abstract

Given a ring with a basis (over some subring), we can ask about the structure constants of the multiplication, i.e. the coefficients when we expand the product of two basis elements in the basis again. The most obvious example is the representation ring of a group (or category), where the coefficients are natural numbers, but we also run into this positivity when taking various cohomology rings of homogeneous spaces, and in either situation we can hope for combinatorial formulae for the coefficients.

I'll start with the most famous and well-understood case of Littlewood-Richardson coefficients, which arise in both ways – the representation ring of Vec, and the ordinary cohomology of Grassmannians – and deserve to have two different rules, one being in terms of "puzzles" (a certain tiling problem). I'll talk about other cohomology theories of Grassmannians and how to compute those with other puzzle pieces. Some of this work is joint with Terry Tao, some with Kevin Purbhoo.

In 1999 I circulated some conjectured puzzle pieces to a few people, for going beyond Grassmannians to richer flag manifolds (with chains of subspaces rather than just one subspace). This turned out to be false for 3-step flag manifolds, but it turns out that the 2-step case is very important, and was proven in 2014 by Buch, Kresch, Purbhoo, and Tamvakis.

Keywords: representation ring, cohomology of Grassmannians, flag manifold.