Graphical Condensation and Plane Partitions

Eric Kuo

University of California, Berkeley

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Plane Partitions

A plane paritition is a finite array of integers such that each row and column is a weakly decreasing sequence of nonnegative integers.

Example:

$$\begin{matrix} 5 & 3 & 2 & 1 \\ 4 & 3 & 1 & 1 \\ 2 & 2 & 1 & \\ 2 & 1 & & \end{matrix}$$

Equivalently, it is a finite subset S of $\mathbb{Z}^+ \times \mathbb{Z}^+ \times \mathbb{Z}^+$ such that if $(x, y, z) \in S$, then $(x', y', z') \in S$ for $x' < x$, $y' < y$, $z' < z$.

Each integer in a plane partition can be represented as a stack of cubes.

MacMahon Polynomials

Let $\mathcal{B}(r,s,t)$ be a box with dimensions $r \times s \times t$.

The MacMahon Polynomial $M(r,s,t)$ is the generating function in which the coefficient of q^n is the number of plane partitions of n cubes that fit in $\mathcal{B}(r,s,t)$.

Example: $M(2,2,2) = 1+q+3q^2+3q^3+4q^4+3q^5+3q^6+q^7+q^8$.

Note that $M(r,s,1)$ is the Gaussian polynomial $\begin{bmatrix} r+s \\ r \end{bmatrix}$.

In 1912, MacMahon published a proof that

$$M(r,s,t) = \prod_{i=1}^{r} \prod_{j=1}^{s} \frac{1-q^{i+j+t-1}}{1-q^{i+j-1}}.$$

Other proofs were published by L. Carlitz (1967) and Gessel and Viennot (1985).

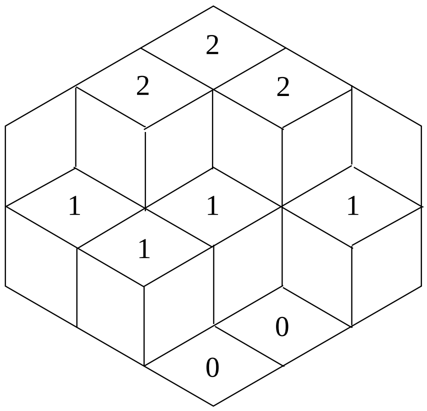
Another way to prove MacMahon's formula:

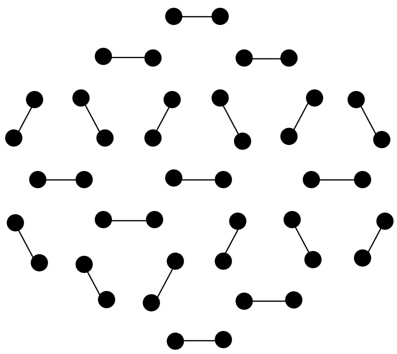
First prove

$$M(r+1,s+1,t)M(r,s,t) = \\ q^t M(r,s+1,t)M(r+1,s,t) \\ +M(r+1,s+1,t-1)M(r,s,t+1).$$

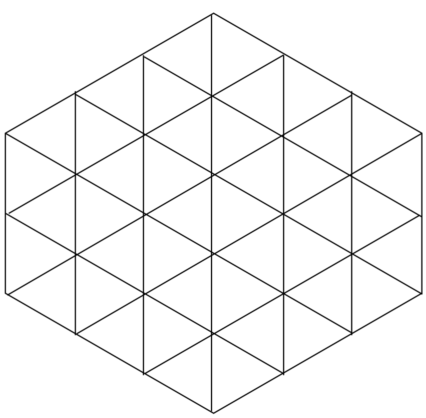
Then prove MacMahon's formula by induction on $r+s+t$, using this identity as the inductive step. (Base cases: $M(r,s,t) = 1$ when $r = 0$, $s = 0$, or $t = 0$.)

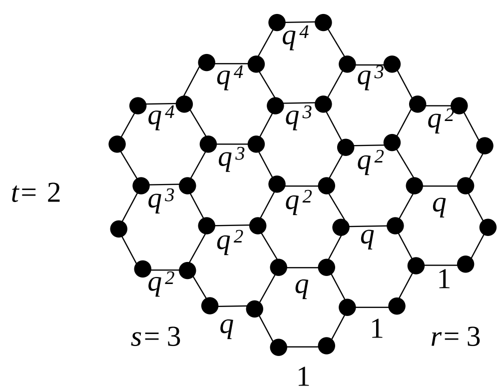
A plane partition in $\mathcal{B}(r, s, t)$ can be represented as a rhombus tiling of a hexagon of sides r, s, t, r, s, t.





Each tiling corresponds to a (perfect) matching in a honeycomb graph.

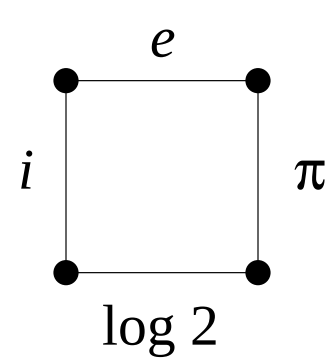




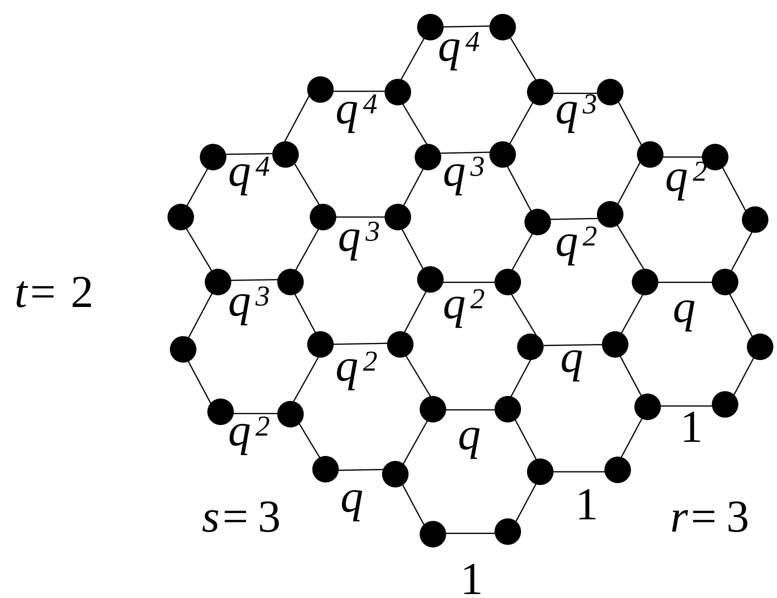
The weight of a matching is the product of the weights of the edges in the matching.

The weighted sum of a graph is the sum of the weights of all possible matchings in a weighted graph.

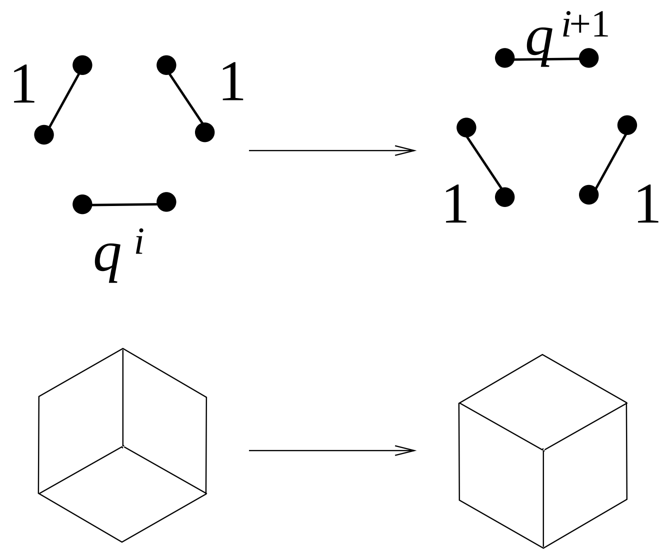
Example: the following graph has weighted sum $e \log 2 + i\pi$.



Consider this weighting scheme for the honey-comb graph:



Adding a cube to a plane partition is like applying these transitions between rhombus tilings or honeycomb matchings:



The matching corresponding to the empty plane partition has weight $q^{rs(s-1)/2}$.

The weighted sum of the weighted honeycomb graph is therefore

$$q^{rs(s-1)/2}M(r,s,t).$$

Graphical Condensation

Graphical condensation is a combinatorial technique that proves identities of the form

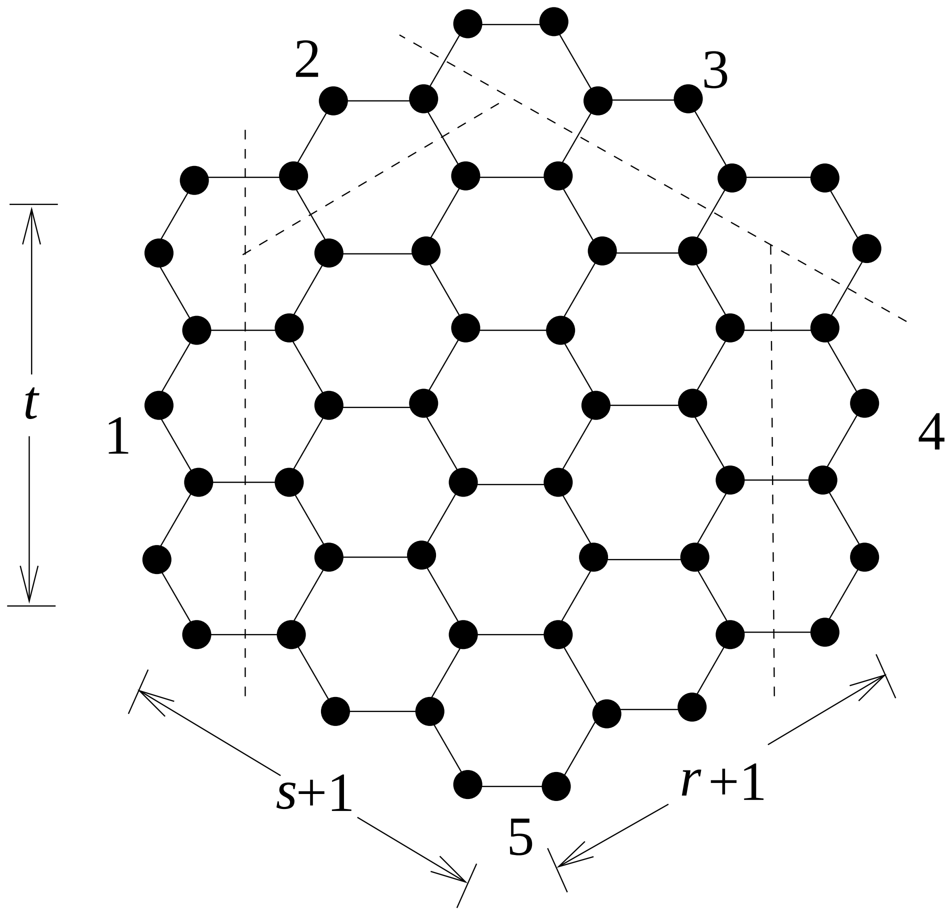
$$M(G_1)M(G_2) = M(G_3)M(G_4) + M(G_5)M(G_6)$$

where $M(G_i)$ is the number of matchings of graph G_i. Each G_i is a planar bipartite graph. More generally, $M(G_i)$ could also be the weighted sum of G_i.

Recall we are trying to prove

$$\begin{aligned} M(r+1, s+1, t)M(r, s, t) = \\ q^t M(r, s+1, t)M(r+1, s, t) \\ + M(r+1, s+1, t-1)M(r, s, t+1). \end{aligned}$$

This Graph G is partitioned into 5 subsets:



Define the subgraphs of G induced by the following subsets:

- H_{12345} = subsets 1,2,3,4,5.
- H_5 = subset 5.
- H_{125} = subsets 1,2,5.
- H_{345} = subsets 3,4,5.
- H_{145} = subsets 1,4,5.
- H_{235} = subsets 2,3,5.

Theorem:

$$M(H_{12345})M(H_5) = \\ M(H_{125})M(H_{345}) + M(H_{145})M(H_{235})$$

Proof:

Superimposing matchings of H_{12345} and H_5 will result in a graph where vertices in subsets 1,2,3,4 have degree 1, and vertices in subset 5 have degree 2.

Same thing happens after combining matchings of H_{125} and H_{345}, or combining matchings of H_{145} with H_{235}.

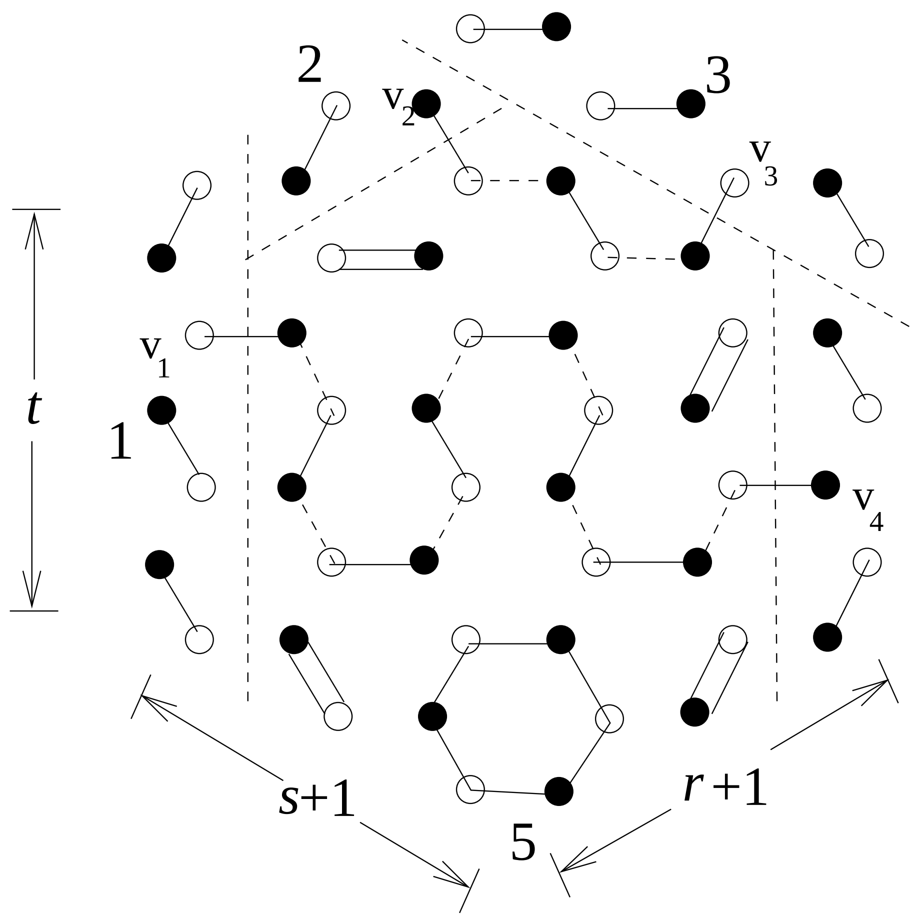
Let H be a multigraph on vertices of G where vertices in subsets 1-4 have degree 1, and vertices in subset 5 have degree 2.

In each of the subsets 1-4, exactly one vertex is joined to a vertex outside its own subset. Call them v_1, v_2, v_3, v_4.

Thus H has two paths running from v_1 to v_2 and v_3 to v_4, or v_1 to v_4 and v_2 to v_3, but not from v_1 to v_3 and v_2 to v_4.

Subset 5 contains cycles doubled edges, and usually the paths among v_1, v_2, v_3, v_4.

Multigraph H



H can always be partitioned into matchings of H_{12345} and H_5.

If v_1 connects to v_2 and v_3 connects to v_4, then H can be partitioned only into matchings of H_{125} and H_{345}.

If v_1 connects to v_4 and v_2 to v_3, then H can be partitioned only into matchings of H_{145} and H_{235}.

Number of possible partitions of H is $2^{k(H)}$ where $k(H)$ is number of cycles in H. Thus

$$\begin{aligned}
&\sum_H 2^{k(H)} W(H) \\
&\quad = M(H_{12345})M(H_5) \\
&\quad = M(H_{125})M(H_{345}) + M(H_{145})M(H_{235}).
\end{aligned}$$

Weighted sums of following graphs:

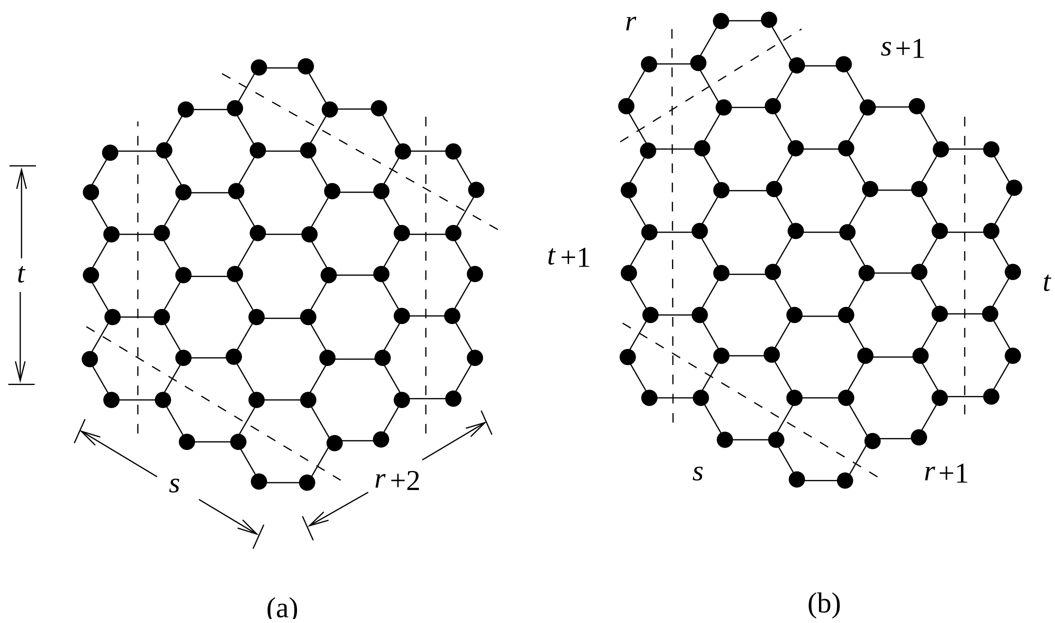
- H_{12345}: $q^{(r+1)(s+1)s/2}M(r+1,s+1,t)$
- H_5: $q^{rs(s-1)/2}M(r,s,t)$
- H_{125}: $q^{r(s+1)s/2}M(r,s+1,t)$
- H_{345}: $q^{s+t}q^{(r+1)s(s-1)/2}M(r+1,s,t)$
- H_{145}: $q^{(r+1)(s+1)s/2}M(r+1,s+1,t-1)$
- H_{235}: $q^{rs(s-1)/2}M(r,s,t+1)$

$$
\begin{aligned}
&\sum_H 2^{k(H)} W(H) \\
&= q^{(r+1)(s+1)s/2} M(r+1,s+1,t) \cdot \\
&\quad q^{rs(s-1)/2} M(r,s,t) \\
&= q^{r(s+1)s/2} M(r,s+1,t) \cdot \\
&\quad q^{s+t} q^{(r+1)s(s-1)/2} M(r+1,s,t) \\
&+ q^{(r+1)(s+1)s/2} M(r+1,s+1,t-1) \cdot \\
&\quad q^{rs(s-1)/2} M(r,s,t+1).
\end{aligned}
$$

Dividing through by $q^{(r+1)(s+1)s/2+rs(s-1)/2}$, we get

$$
\begin{aligned}
&M(r+1,s+1,t)M(r,s,t) = \\
&\quad q^t M(r,s+1,t)M(r+1,s,t) \\
&\quad +M(r+1,s+1,t-1)M(r,s,t+1).
\end{aligned}
$$

Bonus Identities



$$M(r+2,s,t)M(r,s,t) =$$
$$M(r+1,s,t)^2$$
$$-\ q^{r+1}M(r+1,s-1,t+1)M(r+1,s+1,t-1).$$
$$M(r,s,t+1)M(r,s,t) =$$
$$M(r+1,s,t)M(r-1,s,t+1)$$
$$+\ q^{r}M(r,s+1,t)M(r,s-1,t+1).$$