General Measure Spaces.

A. Definition and Examples.

Definition 0.1 A pair $(X, \mathcal{M})$, where X is a set and $\mathcal{M}$ is a σ–algebra of subsets of X, is called a *measurable space.* A function $\mu: \mathcal{M} \to [0, \infty]$ is a *measure* on $(X, \mathcal{M})$ provided that (a) $\mu(\emptyset) = 0$, and (b) if $\{E_n\}_{n=1}^{\infty} \subseteq \mathcal{M}$ is disjoint, then

$$\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n)$$

in which case we say that μ is *countably additive.* The triple $(X, \mathcal{M}, \mu)$ where $(X, \mathcal{M})$ is a measurable space and μ a measure on $(X, \mathcal{M})$ is called a *measure space.*

Examples. (1) $(\mathbf{R}, \mathcal{L}, m)$ where $\mathcal{L}$ is the σ-algebra of Lebesgue measurable subsets of $\mathbf{R}$ and m is Lebesgue measure is a measure space. $(\mathbf{R}, \mathcal{B}, m)$ where $\mathcal{B}$ is the σ-algebra of Borel sets is also a measure space.
(2) Given $f \geq 0$ a measurable function, define for $E \in \mathcal{L}$,

$$\mu(E) = \int_E f.$$

Then $(\mathbf{R}, \mathcal{L}, \mu)$ is a measure space. Can all measures be written in this way?
(3) Let X be any set and let 2^X denote the collection of all subsets of X. Then $(X, 2^X, c)$ is a measure space where c is the *counting measure* defined as follows. $c(E)$ is the number of elements in E if E is finite and $c(E) = \infty$ otherwise.
(4) Let X be any set and let $x_0 \in X$, then $(X, 2^X, \delta_{x_0})$ is a measure space where δ_{x_0} is the *Dirac measure at* x_0 defined as follows. $\delta_{x_0}(E) = 1$ if $x_0 \in E$ and 0 otherwise. Neither counting measure or any Dirac measure on $\mathbf{R}$ corresponds to integration against a measurable function on $\mathbf{R}$.

Remark 0.1 (1) What properties of $(\mathbf{R}, \mathcal{L}, m)$ are common to all measures?

(a) finite additivity.

(b) montonicity.

(c) excision.

(d) countable subadditivity.

(e) continuity of measure.

(f) The Borel-Cantelli Lemma.

(2) There are several important properties of general measures that are not necessarily shared by $(\mathbf{R}, \mathcal{L}, m)$.

Definition 0.2 Let $(X, \mathcal{M}, \mu)$ be a measure space.

(a) μ is *finite* if $\mu(X) < \infty$.

(b) μ is *σ-finite* if X is the union of countably many sets of finite measure. Also any set $E \in \mathcal{M}$ is *σ-finite* if it is the union of countably many sets of finite measure.

(c) $(X, \mathcal{M}, \mu)$ is *complete* if all subsets of any set of measure zero is measurable, that is, is contained in $\mathcal{M}$.

Remark 0.2 (1) $(\mathbf{R}, \mathcal{L}, m)$ is not finite but is σ-finite and complete. $([0,1], \widetilde{\mathcal{L}}, m)$, where $\widetilde{\mathcal{L}} = \{E \in \mathcal{L} \colon E \subseteq [0,1]\}$ is finite (and hence σ-finite) and complete.
(2) $(\mathbf{R}, 2^{\mathbf{R}}, c)$ is not finite, not σ-finite, but is complete. $(\mathbf{R}, 2^{\mathbf{R}}, \delta_{x_0})$ for some $x_0 \in \mathbf{R}$ is finite, σ-finite, and complete.
(3) $(\mathbf{R}, \mathcal{B}, m)$ is not finite, is σ-finite, but is not complete.

B. Signed Measures and the Jordan Decomposition Theorem.

Proposition 0.1 If $(X, \mathcal{M}, \mu_1)$ and $(X, \mathcal{M}, \mu_2)$ are measures, then for any α, $\beta \geq 0$, $\alpha\mu_1 + \beta\mu_2$ is also a measure

Remark 0.3 (1) We have seen that if $f \geq 0$ is any measurable function, then if we define $\mu(E) = \int_E f$, $(\mathbf{R}, \mathcal{L}, \mu)$ is a measure space. What if we dropped the assumption that $f \geq 0$? We still have that $\mu(\emptyset) = 0$ and countable additivity.

(2) However, in order for this set function to be well-defined we must avoid a situation in which $\int_E f$ can fail to be defined. For example, if $f(x) = \sin(x)$ and we want to compute $\mu([0, \infty))$, we would be in trouble since the integral $\int_0^\infty \sin(x)\, dx$ is not well-defined. This is because we can write $[0, \infty) = E_1 \cup E_2$ where E_1 and E_2 are disjoint, $\int_{E_1} \sin(x) = \infty$ and $\int_{E_2} \sin(x) = -\infty$ so that $\int_0^\infty \sin(x) = \infty - \infty$.

(3) We can avoid this problem if we restrict our attention to functions f such that for any set $E \subseteq \mathbf{R}$, $\int_E f$ takes on potentially only one of ∞ or $-\infty$. That is, one of the infinities is always excluded. This concept can be generalized.

Definition 0.3 A *signed measure*, ν, on a measurable space $(X, \mathcal{M})$ is a function $\nu\colon \mathcal{M} \to [-\infty, \infty]$ such that (a) ν assumes at most one of the values ∞ or $-\infty$, (b) $\nu(\emptyset) = 0$, and (c) ν is countably additive.

Remark 0.4 (1) Let f be measurable on $\mathbf{R}$ with the property that $\nu(E) = \int_E f$ defines a signed measure on $\mathbf{R}$. If we let $A = \{x\colon f(x) \geq 0\}$ and $B = \{x\colon f(x) < 0\}$ and define the measures

$$\nu^+(E) = \int_{A\cap E} f = \int_E f^+ \text{ and } \nu^-(E) = -\int_{B\cap E} f = \int_E f^-$$

then the following hold.

(a) The sets A and B satisfy $A \cap B = \emptyset$ and $A \cup B = \mathbf{R}$,

(b) $\nu^+(B) = \nu^-(A) = 0$,

(c) If $E \subseteq A$ then $\nu(E) \geq 0$ and if $E \subseteq B$ then $\nu(E) \leq 0$.

(d) $\nu(E) = \nu^+(E) - \nu^-(E)$.

(2) Each of the properties (a)–(d) can be generalized and is significant in its own right. Taking all four properties together, we say that $\{A, B\}$ is a *Hahn decomposition* of ν.

(3) Such a decomposition always exists.

Definition 0.4 Let $(X, \mathcal{M}, \mu_1)$ and $(X, \mathcal{M}, \mu_2)$ be measures. If there exist sets A and B such that $A \cap B = \emptyset$ and $A \cup B = X$ such that $\nu_1(B) = \nu_2(A) = 0$ then we say that μ_1 and μ_2 are *mutually singular*, sometimes denoted $\mu_1 \perp \mu_2$.

Definition 0.5 Let ν be a signed measure. A measurable set A with the property that for every measurable subset E of A, $\nu(E) \geq 0$ is called a *positive set* for ν, and a measurable set B with the property that for every measurable subset E of B, $\nu(E) \leq 0$ is called a *negative set* for ν. A measurable set C with the property that for every subset E of C, $\nu(E) = 0$ is called a *null set* for ν.

Theorem 0.1 (Hahn Decomposition Theorem) Let ν be a signed measure on the measurable space $(X, \mathcal{M})$. Then there is a positive set A and a negative set B such that $A \cap B = \emptyset$ and $A \cup B = X$.

Claim 1: Every subset of a positive set is positive, and the countable union of positive sets is positive.
Claim 2: (Hahn's Lemma) Let E be a measurable set with $0 < \nu(E) < \infty$. Then there is a measurable subset E_0 of E that is positive and has positive measure.

Theorem 0.2 (Jordan Decomposition Theorem) Let ν be a signed measure on the measurable space $(X, \mathcal{M})$. Then there are two mutually singular measures ν^+ and ν^- on $(X, \mathcal{M})$ such that $\nu = \nu^+ - \nu^-$ and this pair is unique.

Remark 0.5 We call the measure $|\nu| = \nu^+ + \nu^-$ the *total variation measure* associated to ν. Always $|\nu(E)| \leq |\nu|(E)$ but equality does not hold in general. An equivalent definition of $|\nu|$ is the following

$$|\nu|(E) = \sup \sum_{k=1}^{n} |\nu(E_k)|$$

where the supremum is taken over all finite, disjoint collections $\{E_k\}_{k=1}^{n}$ of subsets of E.