General Measure Spaces.

A. Definition and Examples.

Definition 0.1 A pair \((X, \mathcal{M})\), where \(X\) is a set and \(\mathcal{M}\) is a \(\sigma\)-algebra of subsets of \(X\), is called a measurable space. A function \(\mu: \mathcal{M} \to [0, \infty]\) is a measure on \((X, \mathcal{M})\) provided that (a) \(\mu(\emptyset) = 0\), and (b) if \(\{E_n\}_{n=1}^\infty \subseteq \mathcal{M}\) is disjoint, then
\[
\mu \left( \bigcup_{n=1}^\infty E_n \right) = \sum_{n=1}^\infty \mu(E_n)
\]
in which case we say that \(\mu\) is countably additive. The triple \((X, \mathcal{M}, \mu)\) where \((X, \mathcal{M})\) is a measurable space and \(\mu\) a measure on \((X, \mathcal{M})\) is called a measure space.

Examples. (1) \((\mathbb{R}, \mathcal{L}, m)\) where \(\mathcal{L}\) is the \(\sigma\)-algebra of Lebesgue measurable subsets of \(\mathbb{R}\) and \(m\) is Lebesgue measure is a measure space. \((\mathbb{R}, \mathcal{B}, m)\) where \(\mathcal{B}\) is the \(\sigma\)-algebra of Borel sets is also a measure space.
(2) Given \(f \geq 0\) a measurable function, define for \(E \in \mathcal{L}\),
\[
\mu(E) = \int_E f.
\]
Then \((\mathbb{R}, \mathcal{L}, \mu)\) is a measure space. Can all measures be written in this way?
(3) Let \(X\) be any set and let \(2^X\) denote the collection of all subsets of \(X\). Then \((X, 2^X, c)\) is a measure space where \(c\) is the counting measure defined as follows. \(c(E)\) is the number of elements in \(E\) if \(E\) is finite and \(c(E) = \infty\) otherwise.
(4) Let \(X\) be any set and let \(x_0 \in X\), then \((X, 2^X, \delta_{x_0})\) is a measure space where \(\delta_{x_0}\) is the Dirac measure at \(x_0\) defined as follows. \(\delta_{x_0}(E) = 1\) if \(x_0 \in E\) and 0 otherwise. Neither counting measure or any Dirac measure on \(\mathbb{R}\) corresponds to integration against a measurable function on \(\mathbb{R}\).

Remark 0.1 (1) What properties of \((\mathbb{R}, \mathcal{L}, m)\) are common to all measures?

(a) finite additivity.
(b) monotonicity.
(c) excision.
(d) countable subadditivity.
(e) continuity of measure.
(f) The Borel-Cantelli Lemma.
(2) There are several important properties of general measures that are not necessarily shared by \((\mathbb{R}, \mathcal{L}, m)\).

**Definition 0.2** Let \((X, \mathcal{M}, \mu)\) be a measure space.

- (a) \(\mu\) is finite if \(\mu(X) < \infty\).
- (b) \(\mu\) is \(\sigma\)-finite if \(X\) is the union of countably many sets of finite measure. Also any set \(E \in \mathcal{M}\) is \(\sigma\)-finite if it is the union of countably many sets of finite measure.
- (c) \((X, \mathcal{M}, \mu)\) is complete if all subsets of any set of measure zero is measurable, that is, is contained in \(\mathcal{M}\).

**Remark 0.2**

1. \((\mathbb{R}, \mathcal{L}, m)\) is not finite but is \(\sigma\)-finite and complete. \(([0, 1], \tilde{\mathcal{L}}, m)\), where \(\tilde{\mathcal{L}} = \{E \in \mathcal{L}: E \subseteq [0, 1]\}\) is finite (and hence \(\sigma\)-finite) and complete.
2. \((\mathbb{R}, 2^\mathbb{R}, c)\) is not finite, not \(\sigma\)-finite, but is complete. \((\mathbb{R}, 2^\mathbb{R}, \delta_{x_0})\) for some \(x_0 \in \mathbb{R}\) is finite, \(\sigma\)-finite, and complete.
3. \((\mathbb{R}, \mathcal{B}, m)\) is not finite, is \(\sigma\)-finite, but is not complete.
B. Signed Measures and the Jordan Decomposition Theorem.

**Proposition 0.1** If \((X, \mathcal{M}, \mu_1)\) and \((X, \mathcal{M}, \mu_2)\) are measures, then for any \(\alpha, \beta \geq 0\), \(\alpha \mu_1 + \beta \mu_2\) is also a measure.

**Remark 0.3**

1. We have seen that if \(f \geq 0\) is any measurable function, then if we define \(\mu(E) = \int_E f\), \((R, \mathcal{L}, \mu)\) is a measure space. What if we dropped the assumption that \(f \geq 0\)? We still have that \(\mu(\emptyset) = 0\) and countable additivity.

2. However, in order for this set function to be well-defined we must avoid a situation in which \(\int_E f\) can fail to be defined. For example, if \(f(x) = \sin(x)\) and we want to compute \(\mu([0, \infty))\), we would be in trouble since the integral \(\int_0^\infty \sin(x)\) is not well-defined. This is because we can write \([0, \infty) = E_1 \cup E_2\) where \(E_1\) and \(E_2\) are disjoint, \(\int_{E_1} \sin(x) = \infty\) and \(\int_{E_2} \sin(x) = -\infty\) so that \(\int_0^\infty \sin(x) = \infty - \infty\).

3. We can avoid this problem if we restrict our attention to functions \(f\) such that for any set \(E \subseteq R\), \(\int_E f\) takes on potentially only one of \(\infty\) or \(-\infty\). That is, one of the infinities is always excluded. This concept can be generalized.

**Definition 0.3** A **signed measure**, \(\nu\), on a measurable space \((X, \mathcal{M})\) is a function \(\nu: \mathcal{M} \to [-\infty, \infty]\) such that (a) \(\nu\) assumes at most one of the values \(\infty\) or \(-\infty\), (b) \(\nu(\emptyset) = 0\), and (c) \(\nu\) is countably additive.

**Remark 0.4**

1. Let \(f\) be measurable on \(R\) with the property that \(\nu(E) = \int_E f\) defines a signed measure on \(R\). If we let \(A = \{x: f(x) \geq 0\}\) and \(B = \{x: f(x) < 0\}\) and define the measures

\[
\nu^+(E) = \int_{A \cap E} f = \int_E f^+ \quad \text{and} \quad \nu^-(E) = -\int_{B \cap E} f = \int_E f^-
\]

then the following hold.

(a) The sets \(A\) and \(B\) satisfy \(A \cap B = \emptyset\) and \(A \cup B = R\),

(b) \(\nu^+(B) = \nu^-(A) = 0\),

(c) If \(E \subseteq A\) then \(\nu(E) \geq 0\) and if \(E \subseteq B\) then \(\nu(E) \leq 0\).

(d) \(\nu(E) = \nu^+(E) - \nu^-(E)\).

2. Each of the properties (a)–(d) can be generalized and is significant in its own right. Taking all four properties together, we say that \(\{A, B\}\) is a **Hahn decomposition** of \(\nu\).

3. Such a decomposition always exists.

**Definition 0.4** Let \((X, \mathcal{M}, \mu_1)\) and \((X, \mathcal{M}, \mu_2)\) be measures. If there exist sets \(A\) and \(B\) such that \(A \cap B = \emptyset\) and \(A \cup B = X\) such that \(\nu_1(B) = \nu_2(A) = 0\) then we say that \(\mu_1\) and \(\mu_2\) are **mutually singular**, sometimes denoted \(\mu_1 \perp \mu_2\).
**Definition 0.5** Let \( \nu \) be a signed measure. A measurable set \( A \) with the property that for every measurable subset \( E \) of \( A \), \( \nu(E) \geq 0 \) is called a **positive set** for \( \nu \), and a measurable set \( B \) with the property that for every measurable subset \( E \) of \( B \), \( \nu(E) \leq 0 \) is called a **negative set** for \( \nu \). A measurable set \( C \) with the property that for every subset \( E \) of \( C \), \( \nu(E) = 0 \) is called a **null set** for \( \nu \).

**Theorem 0.1** (Hahn Decomposition Theorem) Let \( \nu \) be a signed measure on the measurable space \((X, \mathcal{M})\). Then there is a positive set \( A \) and a negative set \( B \) such that \( A \cap B = \emptyset \) and \( A \cup B = X \).

**Claim 1:** Every subset of a positive set is positive, and the countable union of positive sets is positive.

**Claim 2:** (Hahn’s Lemma) Let \( E \) be a measurable set with \( 0 < \nu(E) < \infty \). Then there is a measurable subset \( E_0 \) of \( E \) that is positive and has positive measure.

**Theorem 0.2** (Jordan Decomposition Theorem) Let \( \nu \) be a signed measure on the measurable space \((X, \mathcal{M})\). Then there are two mutually singular measures \( \nu^+ \) and \( \nu^- \) on \((X, \mathcal{M})\) such that \( \nu = \nu^+ - \nu^- \) and this pair is unique.

**Remark 0.5** We call the measure \( |\nu| = \nu^+ + \nu^- \) the **total variation measure** associated to \( \nu \). Always \( |\nu(E)| \leq |\nu|(E) \) but equality does not hold in general. An equivalent definition of \( |\nu| \) is the following

\[
|\nu|(E) = \sup \sum_{k=1}^{n} |\nu(E_k)|
\]

where the supremum is taken over all finite, disjoint collections \( \{E_k\}_{k=1}^{n} \) of subsets of \( E \).