The Vitali Covering Lemma and the Lebesgue Differentiation Theorem.

A. Monotone Functions.

Definition 0.1 A collection $\mathcal{F}$ of closed, bounded, nondegenerate intervals is said to cover a set E *in the sense of Vitali* if for each $x \in E$ and $\epsilon > 0$ there is an $I \in \mathcal{F}$ containing x with $m(I) < \epsilon$.

Theorem 0.1 Let f be monotone on (a, b) (not necessarily finite). Then f is continuous except possibly at a countable number of points in (a, b).

Lemma 0.1 (Vitali Covering Lemma) Let E be a set of finite outer measure and let $\mathcal{F}$ be a Vitali covering of E. Then given $\delta > 0$ there is a finite, disjoint collection $\{I_k\}_{k=1}^n \subseteq \mathcal{F}$ such that

$$\sum_{k=1}^{n} m(I_k) \geq m^*(E) - \delta.$$

Corollary 0.1 In the situation of the Vitali Covering Lemma, the intervals can be chosen in such a way that

$$m^*\left(E - \bigcup_{i=1}^{N} I_i\right) < 2\delta.$$

B. The Lebesgue Differentiation Theorem.

Theorem 0.2 (Lebesgue Differentiation Theorem) Let f be monotone on (a, b). Then f is differentiable a.e. in (a, b).

Definition 0.2 In order to describe differentiability, we define for a given f the *upper and lower derivatives* by

$$\overline{D}f(x) = \limsup_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0^+} \sup_{0<|t|\leq h} \frac{f(x+t) - f(x)}{t} = \inf_{h>0} \sup_{0<|t|\leq h} \frac{f(x+t) - f(x)}{t},$$

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In order to prove the Lebesgue Differentiation Theorem, we need the following Lemma.

Lemma 0.2 (Mean Value Theorem) Let f be increasing on $[a, b]$, a finite closed interval. Then for each $\alpha > 0$,

$$m^*(\{x \in (a, b) \colon \overline{D}f(x) \geq \alpha\}) \leq \frac{1}{\alpha}[f(b) - f(a)]$$

and

$$m^*(\{x \in (a, b) \colon \overline{D}f(x) = \infty\}) = 0.$$

Corollary 0.2 If f is increasing on $[a, b]$ then f' is integrable on $[a, b]$ and

$$\int_a^b f' \leq f(b) - f(a).$$