# Fundamental Theorem of Calculus Part 1.

**Remark 0.1** (1) There are two parts to the Fundamental Theorem that we consider separately.

(a) $\dfrac{d}{dx}\displaystyle\int_a^x f(y)\,dy = f(x)$, and

(b) $\displaystyle\int_a^b f'(y)\,dy = f(b) - f(a)$.

(2) Note that if $f$ is continuous on $\mathbf{R}$ then (a) holds and if $f$ is continuously differentiable on $\mathbf{R}$ then (b) holds. Do these identities hold in greater generality? And in what sense do they hold?

(3) Recall that the Cantor function $\varphi(x)$ satisfies $\varphi'(x) = 0$ a.e. on $[0, 1]$, and $\varphi$ is continuous, nondecreasing, and $\varphi(0) = 0$ and $\varphi(1) = 1$.

(4) Hence if $f = \varphi'$ then (a) fails in the sense that the integral on the left side is defined and equals zero for every $x$, so that its derivative is identically zero, but the right side is zero only almost everywhere, and in fact fails to be zero on an uncountable set. Also, if $f = \varphi$ then (b) fails for $a = 0$ and $b = 1$.

## A. The maximal function.

**Definition 0.1** Suppose $f$ is integrable on $\mathbf{R}$. We define its *maximal function* $f^*(x)$ by

$$f^*(x) = \sup_{x\in I} \frac{1}{m(I)} \int_I |f|$$

where the supremum is taken over all intervals $I$ containing $x$.

**Remark 0.2** (1) Note that

$$\frac{d}{dx}\int_a^x f = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f = \lim_{h\to 0} \frac{1}{m(I_h)}\int_{I_h} f$$

where $I_h = (x, x + h)$ or $(x + h, x)$.

(2) For each $h$ $\left|\dfrac{1}{m(I_h)}\displaystyle\int_{I_h} f\right| \le f^*(x)$ and we would like to consider the relationship between $f(x)$ and $f^*(x)$.

**Proposition 0.1** If $f$ is integrable on $\mathbf{R}$ then

(a) $f^*$ is measurable,

(b) $f^*$ is integrable, and

(c) For all $\alpha > 0$, $m(\{x: f^*(x) > \alpha\}) \leq (3/\alpha)\|f\|_1$.

**B. The Lebesgue set.**

**Theorem 0.1** (Lebesgue) If $f$ is integrable on $\mathbf{R}$ then for a.e. $x$,

$$\lim_{\substack{m(I)\to 0 \\ x\in I}} \frac{1}{m(I)} \int_I f = f(x).$$

**Remark 0.3** (1) Since we are only interested in the averages of $f$ over small intervals, it suffices to assume that $f$ is only *locally integrable*, that is, for every finite interval $I$, $\int_I |f| < \infty$.
(2) This allows us to include functions that do not go to zero at infinity, such as polynomials, in our discussion.

**Definition 0.2** Suppose $f$ is locally integrable on $\mathbf{R}$. We define the *Lebesgue set* of $f$ to be the set of all $x \in \mathbf{R}$ such that

$$\lim_{\substack{m(I)\to 0 \\ x\in I}} \frac{1}{m(I)} \int_I |f(y) - f(x)|\, dy = 0.$$

**Remark 0.4** If $x$ is a point of continuity of $f$, then $x$ is in the Lebesgue set of $f$, and if $x$ is in the Lebesgue set of $f$ then

$$\lim_{\substack{m(I)\to 0 \\ x\in I}} \frac{1}{m(I)} \int_I f = f(x).$$

**Theorem 0.2** (Lebesgue) If $f$ is locally integrable on $\mathbf{R}$ then a.e. $x \in \mathbf{R}$ is in the Lebesgue set of $f$.

**Remark 0.5** Note that

$$\lim_{\substack{m(I)\to 0 \\ x\in I}} \frac{1}{m(I)} \int_I f = f(x)$$

allows $I$ to shrink to $x$ in a variety of ways. In particular, if $f$ is locally integrable on $\mathbf{R}$ then

$$\frac{d}{dx} \int_a^x f = \lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f = f(x)$$

for a.e. $x$. This means that if $f$ is locally integrable then $\int_a^x f$ is differentiable a.e. but not necessarily everywhere, as the example of the Cantor function illustrates.