Convergence in measure.

A. Definition and Basic Properties.

Definition 0.1 Let $\{f_n\}$ be a sequence of measurable functions, finite a.e., on a set E, and let f be a measurable function, finite a.e., on E. Then $f_n \to f$ *in measure* on E provided that for every $\eta > 0$,

$$\lim_{n\to\infty} m(\{x \in E \colon |f_n(x) - f(x)| > \eta\}) = 0.$$

Remark 0.1 We assume as part of the definition that f_n and f are finite a.e. in order that the difference $|f_n(x) - f(x)|$ makes sense a.e., and we assume that f_n and f are measurable so that the sets defined in the definition are guaranteed to be measurable.

Proposition 0.1 If $m(E) < \infty$ and $f_n \to f$ pointwise a.e. on E with f finite a.e., then $f_n \to f$ in measure on E.

Remark 0.2 (1) What happens to the above proposition if $m(E) = \infty$?

(2) Does the converse hold? That is, if $f_n \to f$ in measure does it imply pointwise a.e. convergence?

(3) We define the *dyadic subintervals* of $[0, 1]$ as follows: Let $I_{j,k} = [2^{-j}k, 2^{-j}(k + 1)]$ for integers $j \geq 0$ and $k = 0, 1, \ldots, 2^j - 1$. For such j and k we define the injective mapping $(j, k) \mapsto n = 2^j + k$ which is onto the natural numbers. If we define $f_n = \chi_{I_{j,k}}$ where n corresponds to the pair (j, k), then $f_n \to 0$ in measure but f_n does not converge to anything pointwise in the sense that for every $x \in [0, 1]$ the sequence $\{f_n(x)\}$ does not converge. Hence the converse of the proposition fails to hold.

Theorem 0.1 (Riesz) Suppose that $f_n \to f$ in measure on E. Then there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.e. on E.

B. Convergence theorems for convergence in measure.

Theorem 0.2 (Vitali Theorem) Let f_n be a sequence of non-negative integrable functions on E. Then $\int_E f_n \to 0$ as $n \to \infty$ if and only if $f_n \to 0$ in measure and $\{f_n\}$ is uniformly integrable and tight over E.

Theorem 0.3 (Fatou's Lemma) Let f_n be a sequence of non-negative measurable functions on E. If $f_n \to f$ in measure on E then

$$\int_E f \leq \liminf_{n\to\infty} \int_E f_n.$$

Remark 0.3 (1) The previous proof of Fatou's Lemma can be used, but there is a point in the proof where we invoke the Bounded Convergence Theorem. The proof of the BCT uses Egoroff's Theorem which we do not have for convergence in measure. Do we have the BCT for convergence in measure?

(2) Once Fatou's Lemma has been established for convergence in measure the other main convergence theorems, Monotone Convergence Theorem, Dominated Convergence Theorem also hold. You should check whether or not the proofs in these cases go through for convergence in measure.

C. The space $L^1(\mathbf{R})$.

Definition 0.2 For f integrable on $\mathbf{R}$, we define the L^1–*norm* of f to be $\|f\|_1 = \int_{\mathbf{R}} |f|$.

Remark 0.4 (1) We can extend the notion of measurable and integrable function to complex valued functions without too much difficulty. Let f be a function from $\mathbf{R}$ to $\mathbf{C}$, the complex plane. Then we can write $f = u + iv$ where u and v are real-valued functions.

(2) It is true that f is measurable if and only if both u and v are measurable. Here f measurable means that the inverse image of open sets in $\mathbf{C}$ is measurable.

(3) It is also true that f is integrable if and only if both u and v are integrable. In this context, integrable means that $\int_{\mathbf{R}} |f| < \infty$. Note that $|f|$ is a real valued, nonnegative function on $\mathbf{R}$. This result follows from the observation that $\max\{|u|, |v|\} \leq |f| \leq |u| + |v|$.

Remark 0.5 (1) It is not hard to see that $\|\cdot\|_1$ has some of the properties of a norm, that is, for f and g integrable,

(a) $\|\alpha f\|_1 = |\alpha| \|f\|_1$, for all $\alpha \in \mathbf{R}$.

(b) $\|f + g\|_1 \leq \|f\|_1 + \|g\|_1$.

(c) $\|f\|_1 = 0 \iff f = 0,\ a.e.$

We can define an associated distance function $d(f,g) = \|f - g\|_1$ on the set of all functions integrable on $\mathbf{R}$, and that this distance function has some of the properties of a metric.

(2) Note that if we look at the collection of all complex-valued functions integrable on $\mathbf{R}$ then this set forms a vector space under addition of functions and scalar multiplication by complex numbers.

(3) So we would like to say that the space of functions integrable on $\mathbf{R}$ with the L^1 norm forms a normed linear space. However this is not quite correct. The problem is that $\|f\|_1 = 0$ implies that $f = 0$ a.e., not that f is identically zero. Hence if $f \neq g$ on a set of measure zero, then $\|f - g\|_1 = 0$ so that as far as the L^1 norm is concerned, f and g are indistinguishable.

(4) So if we define the relation $\sim$ on the space of integrable functions by $f \sim g$ if and only if $\|f - g\|_1 = 0$ then this defines an equivalence relation on the space of integrable functions.

Definition 0.3 We define $L^1(\mathbf{R})$ to be the set of equivalence classes of functions integrable on $\mathbf{R}$. In this case, the L^1-norm defines a norm, and $d(f,g)$ a metric on $L^1(\mathbf{R})$. It follows from this that $L^1(\mathbf{R})$ is a normed linear space over $\mathbf{C}$.

Definition 0.4 Recall that a normed linear space is said to be *complete* if every Cauchy sequence in the space converges to an element of the space. Recall also that a set is *dense* in a normed linear space if every element of the space can be approximated arbitrarily closely by an element of the set.

Theorem 0.4 (Riesz-Fischer Theorem) $L^1(\mathbf{R})$ is complete.

Theorem 0.5 The following families of functions are dense in $L^1(\mathbf{R})$.

(a) Simple functions.

(b) Step functions.

(c) Continuous functions with finite support

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