Vitali's Convergence Theorems.

Consider the central hypothesis in the Lebesgue Dominated Convergence Theorem, namely that there is a function g integrable on E such that for all n, $|f_n| \leq g$ on E. This hypothesis implies two properties of $\{f_n\}$ that are important in their own right.

A. Uniform Integrability.

Proposition 0.1 Let f be integrable over E. Then for every $\epsilon > 0$ there is a $\delta > 0$ such that if $A \subseteq E$ and $m(A) < \delta$ then $\int_E |f| < \epsilon$.

Definition 0.1 A family $\mathcal{F}$ of measurable functions on E is *uniformly integrable* over E if given $\epsilon > 0$ there is a $\delta > 0$ such that if $A \subseteq E$ and $m(A) < \delta$ then $\int_E |f| < \epsilon$ for all $f \in \mathcal{F}$.

Remark 0.1 (1) Note that the only assumption about A in both the proposition and the definition is that $m(A) < \delta$. That is, the inequality holds for any subset A as long as $m(A) < \delta$.
(2) Note that if the sequence f_n satisfies $|f_n| \leq g$ for some integrable function g on E, then the family $\{f_n\}$ is uniformly integrable over E.

Theorem 0.1 (Vitali) Let $m(E) < \infty$ and suppose that the sequence $\{f_n\}$ is uniformly integrable over E. If $f_n \to f$ pointwise a.e. on E, then f is integrable on E and $\int_E f_n \to \int_E f$ as $n \to \infty$.

Theorem 0.2 (Vitali converse) Let $m(E) < \infty$ and suppose that $\{h_n\}$ is a sequence of non-negative integrable functions that converge pointwise a.e. on E to 0. Then $\lim_n \int_E h_n = 0$ if and only if $\{h_n\}$ is uniformly integrable over E.

B. Tightness.

Proposition 0.2 Let f be integrable over E. Then for every $\epsilon > 0$ there is a subset $E_0 \subseteq E$ with $m(E_0) < \infty$ such that $\int_{E-E_0} |f| < \epsilon$.

Definition 0.2 A family $\mathcal{F}$ of measurable functions on E is *tight* over E if given $\epsilon > 0$ there is a subset $E_0 \subseteq E$ with $m(E_0) < \infty$ such that $\int_{E-E_0} |f| < \epsilon$ for all $f \in \mathcal{F}$.

Remark 0.2 Note that if the sequence f_n satisfies $|f_n| \leq g$ for some integrable function g on E, then the family $\{f_n\}$ is tight over E.

Theorem 0.3 (Vitali) Let $\{f_n\}$ be a sequence that is uniformly integrable and tight over E. If $f_n \to f$ pointwise a.e. on E, then f is integrable on E and $\int_E f_n \to \int_E f$ as $n \to \infty$.

Theorem 0.4 (Vitali converse) Let $\{h_n\}$ be a sequence of non-negative integrable functions on E that converge pointwise a.e. on E to 0. Then $\lim_n \int_E h_n = 0$ if and only if $\{h_n\}$ is uniformly integrable and tight over E.