# Convergence Theorems.

## A. Lebesgue integral of a nonnegative function.

**Definition 0.1** Let $f \geq 0$ on a measurable set $E$. Then

$$\int_E f = \sup \left\{ \int_E h \colon h \text{ bounded, measurable, finite support, } 0 \leq h \leq f \right\}.$$

**Remark 0.1** (1) We define the *support* of a measurable function $f$ on $E$ by $\{x \in E \colon f(x) \neq 0\}$. We say that $f$ has *finite support* if $m(\{x \in E \colon f(x) \neq 0\}) < \infty$.

(2) The integral defined above exists as an extended real number for any nonnegative measurable function $f$ defined on any measurable set $E$. So for example, by this definition, $\int_{\mathbf{R}} 1 = \infty$.

(3) The integral defined above is linear and monotone for nonnegative measurable functions. The proof of this is very similar to the proof for bounded functions on sets of finite measure.

## B. The General Lebesgue integral.

**Definition 0.2** A measurable function $f$ on a set $E$ is *integrable* on $E$ if

$$\int_E |f| < \infty$$

where the integral of the nonnegative function $|f|$ is defined above. In this case, we define

$$\int_E f = \int_E f^+ - \int_E f^-.$$

**Remark 0.2** (1) Recall that $f^+ = \max\{f, 0\}$ is the positive part of $f$ and that $f^- = -\min\{f, 0\} = (-f)^+$ is the negative part. Hence $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

(2) Note that if $\int_E |f| < \infty$ then by monotonicity, $\int_E f^+ < \infty$ and $\int_E f^- < \infty$.

(3) Note that always, $|\int_E f| \leq \int_E |f|$ if $f$ is integrable on $E$.

**Theorem 0.1** The general Lebesgue integral $\int_E$ is linear and monotone for integrable functions on $E$.

**C. Convergence theorems.**

**Theorem 0.2** (Fatou's Lemma) Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $E$ and suppose that $f_n \to f$ pointwise a.e. on $E$. Then

$$\int_E f \leq \liminf \int_E f_n.$$

**Theorem 0.3** (Monotone Convergence Theorem) Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on $E$ and suppose that $f_n \to f$ pointwise a.e. on $E$. Then

$$\int_E f = \lim_n \int_E f_n.$$

**Theorem 0.4** (Beppo Levi's Lemma) Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on $E$ and suppose that the sequence of real numbers $\{\int_E f_n\}$ is bounded. Then $f_n$ converges pointwise on $E$ to a measurable function $f$ that is finite a.e, and furthermore

$$\int_E f = \lim_n \int_E f_n.$$

**Theorem 0.5** (The Lebesgue Dominated Convergence Theorem) Let $\{f_n\}$ be a sequence of measurable functions on $E$ that converge pointwise a.e. on $E$ to the function $f$. Suppose that there is a function $g$, integrable on $E$, such that $|f_n| \leq g$ on $E$ for all $n$. Then $f$ is also integrable on $E$ and

$$\int_E f = \lim_n \int_E f_n.$$

**Corollary 0.1** Suppose that $f$ is integrable on $E$ and that we can write $E = \cup_{n=1}^{\infty} E_n$ where $\{E_n\}$ is a disjoint sequence of measurable sets. Then

$$\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f.$$