The Lebesgue Integral.

A. Bounded functions on sets of finite measure.

Remark 0.1 (1) An alternate definition of the Riemann integral and Riemann integrability is the following: *A bounded function f defined on an interval $[a, b]$ is Riemann integrable provided that*

$$\sup\left\{\int_a^b \varphi\colon \varphi \text{ a step function, } \varphi \le f\right\} = \inf\left\{\int_a^b \psi\colon \psi \text{ a step function, } \psi \ge f\right\}.$$

In this case the common value is the Riemann integral of f.

(2) Letting $f(x) = \chi_{\mathbf{Q}}$, f is not Riemann integrable on $[0, 1]$.

(3) Let $\mathbf{Q} = \{x_1, x_2, \ldots\}$ and define $f_n(x) = \chi_{\{x_1, \ldots, x_n\}}(x)$. Then $f_n \to f$ pointwise on $[0, 1]$, each f_n is Riemann integrable, but f is not Riemann integrable.

Definition 0.1 Let $m(E) < \infty$ and let ψ be a simple function on E. Then the *Lebesgue integral* of ψ is defined by

$$\int_E \psi = \sum_{i=1}^n a_i\, m(E_i)$$

where $\psi = \sum_{i=1}^n a_i \chi_{E_i}$ is the canonical representation of ψ.

Remark 0.2 (1) Using the above definition, the Lebesgue integral is (a) linear and (b) monotonic for simple functions.

(2) From this it follows that the Lebesgue integral is independent of how you represent a simple function.

(3) According to the definition, if f is the characteristic function of the rational numbers, then $\int_E f = 1 \cdot m(\mathbf{Q}) = 0$ and in particular this function is Lebesgue integrable.

Definition 0.2 Let f be a bounded measurable function on a set E with $m(E) < \infty$. We define the *upper* and *lower Lebesgue integral* of f by

$$\sup\left\{\int_E \varphi\colon \varphi \text{ a simple function, } \varphi \le f\right\}, \quad \inf\left\{\int_E \psi\colon \psi \text{ a simple function, } \psi \ge f\right\}$$

respectively. In this case the common value is the Riemann integral of f.

Proposition 0.1 The Lebesgue integral generalizes the Riemann integral in the sense that if f is Riemann integrable, then it is also Lebesgue integrable and the integrals are the same.

Theorem 0.1 Any bounded, measurable function on a set E with $m(E) < \infty$ is Lebesgue integrable on E.

Theorem 0.2 The Lebesgue integral is (a) linear and (b) monotone on sets of finite measure.

B. Convergence results for the Lebesgue integral.

Remark 0.3 (1) If $\{f_n\}$ is a sequence of measurable functions on E with $m(E) < \infty$ and if $f_n \to f$ uniformly on E, then

$$\lim_{n\to\infty} \int_E f_n = \int_E \lim_{n\to\infty} f_n = \int_E f.$$

(2) However, if under the same assumptions, $f_n \to f$ only pointwise, then it need not be the case that $\lim_n \int_E f_n = \int_E f$. There are many counterexamples.

Theorem 0.3 (Bounded Convergence Theorem) Let $\{f_n\}$ be a uniformly bounded sequence of measurable functions on E with $m(E) < \infty$. If $f_n \to f$ pointwise on E then

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$