Littlewood's Three Principles.

A. Limits of sequences of functions.

Definition 0.1 Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions defined on a set E, f a function defined on E, and A a subset of E.

(a) We say that f_n converges to f pointwise on A, denoted $f_n \to f$ pointwise, provided that for every $x \in A$, $f_n(x) \to f(x)$ as a sequence of numbers.

(b) We say that f_n converges to f pointwise almost everywhere on A, denoted $f_n \to f$ a.e., provided that there is a set $B \subseteq A$ such that $f_n \to f$ pointwise on B and $m(A - B) = 0$.

(c) We say that f_n converges to f uniformly on A provided that $\sup_{x \in A} |f_n(x) - f(x)| \to 0$ as $n \to \infty$.

Proposition 0.1 Suppose that f_n is a sequence of measurable functions on a set E and that $f_n \to f$ a.e. on E. Then f is measurable.

Remark 0.1 (1) Replacing "measurable" with "continuous" in the above proposition makes it false.

(2) Replacing "measurable" with "Riemann integrable" in the above proposition makes it false.

(3) Since uniform convergence implies pointwise convergence, the proposition is still true if "$f_n \to f$ a.e." is replaced with "$f_n \to f$ uniformly."

Definition 0.2 A function f is called a *simple function* provided that it is measurable and takes on only finitely many values. If f is simple then there exists a finite collection $\{E_k\}_{k=1}^{n}$ of measurable sets, and numbers $c_1, c_2, \ldots, c_n$ such that $f(x) = \sum_{k=1}^{n} c_k \chi_{E_k}$, where χ_A denotes the characteristic or indicator function of the set A. If for each k, $E_k = f^{-1}(\{c_k\})$, then the above sum is called the *canonical representation* of f.

Proposition 0.2 Let f be a measurable, bounded, real-valued function on E. Then given $\epsilon > 0$, there are simple functions φ_ϵ and ψ_ϵ defined on E with the property that, on E, $\varphi_\epsilon \le f \le \psi_\epsilon$ and $0 \le \psi_\epsilon - \varphi_\epsilon < \epsilon$.

Proposition 0.3 An extended real-valued function f defined on a measurable set E is measurable if and only if there is a sequence of simple functions $\{\varphi_n\}$ defined on E such that $\varphi_n \to f$ pointwise on E and where for all n $|\varphi_n| \le |f|$ on E.

B. Littlewood's Principles.

Remark 0.2 (1) The three principles are:

- Every measurable set is *nearly* the union of a finite collection of disjoint open intervals.
- Every measurable function is *nearly* continuous.
- Every pointwise convergent sequence of functions is nearly uniformly convergent.

(2) We have seen already the first principle in the result that says: *If E is a measurable set with finite measure then for every $\epsilon > 0$ there is a collection $\{I_k\}_{k=1}^n$ of disjoint, open intervals such that if $\mathcal{O} = \cup_{k=1}^n I_k$ then $m(E \Delta \mathcal{O}) < \epsilon$.*

(3) The other principles have the same flavor in the sense that there is a set of arbitrarily small measure such that the desirable property is realized off that set.

Theorem 0.1 (Egoroff's Theorem) Let E be a set of finite measure, and $\{f_n\}$ a sequence of measurable functions on E such that $f_n \to f$ pointwise on E. Then given $\epsilon > 0$, there is a closed set F with $F \subseteq E$ such that $f_n \to f$ uniformly on F and $m(E - F) < \epsilon$.

Theorem 0.2 (Lusin's Theorem) Let f be a real-valued, measurable function defined on a set E. Then given $\epsilon > 0$ there is a function g continuous on $\mathbf{R}$, and a closed set F with $F \subseteq E$ such that $f = g$ on F and $m(E - F) < \epsilon$.