Measurable Functions.

A. Considerations of the Riemann Integral.

Recall the definition of the Riemann integral $\int_a^b f(x)\,dx$.

- Partition the interval $[a, b]$ as $P\colon a = x_0 < x_1 < x_2 < \cdots < x_n = b$.
- Define m_i to be the infimum, and M_i the supremum, of f on the ith subinterval in the partition.
- Form the sums $U(f, P) = \sum_{i=1}^{n} M_i(x_i - x_{i-1})$ and $L(f, P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1})$.
- If $\inf_P U(f, P) = \sup_Q L(f, Q)$ where the inf and the sup are taken over all partitions P and Q of $[a, b]$ then this common value is the value of the Riemann integral.

There are some pretty obvious problems with the Riemann integral.

- There are fairly easily constructed functions that are not Riemann integrable.
- It is possible to have f_n integrable for each n, $f_n \to f$ pointwise, but f not Riemann integrable.
- In the definition of the Riemann integral we are essentially approximating $f(x)$ from above and below by *step functions*, and hoping that the integrals of these step functions can be used to approximate the integral of $f(x)$.

The Lebesgue approach.

- Rather than partition the domain of integration $[a, b]$, partition the range of the function $f([a, b])$ using some partition P.
- Then the collection of sets $f^{-1}(P)$ partitions the domain $[a, b]$ into disjoint sets, not necessarily intervals.
- We approximate $f(x)$ by functions that are constant on the "induced partition" of $[a, b]$. Such functions are called *simple functions.*
- These approximations are more versatile because the partition takes into account the structure of the function and not just the structure of the domain of integration.
- At a minimum, the sets in this induced partition of $[a, b]$ must be measurable, and leads us to our definition of measurable function.

B. Measurable Functions.

Definition 0.1 The real-valued function f defined on $E \subseteq \mathbf{R}$ is *measurable* if for all $c \in \mathbf{R}$,

$$\{x \in E \colon f(x) > c\} = f^{-1}((c, \infty))$$

is measurable.

Proposition 0.1 In the above definition, we can replace $\{x \in E \colon f(x) > c\}$ by $\{x \in E \colon f(x) \geq c\}$, $\{x \in E \colon f(x) < c\}$, or $\{x \in E \colon f(x) \leq c\}$.

Proposition 0.2 f is measurable if and only if for every open set $\mathcal{O} \subseteq \mathbf{R}$, $f^{-1}(\mathcal{O})$ is measurable. From this it follows that if f is continuous on its measurable domain, then f is measurable.

Proposition 0.3

- The modification of a measurable function on a set of measure zero is measurable.
- The restriction of a measurable function to a measurable subset of its domain is measurable.
- If f and g are measurable with disjoint domains, then $f \cup g$ is measurable.
- Linear combinations of finite collections of measurable functions, each of which is finite almost everywhere, are measurable.
- Products of finite collections of measurable functions, each of which is finite almost everywhere, are measurable.
- If g is measurable and f is continuous, then $f \circ g$ is measurable.
- If f is measurable, then so is its positive part f^+ and its negative part f^-. Consequently $|f|$ is also measurable.

Examples:
(1) If f and g are measurable, then $f \circ g$ need not be measurable.
(2) If f is measurable and g is continuous, then $f \circ g$ need not be measurable.