Measurable Functions.

A. Considerations of the Riemann Integral.

Recall the definition of the Riemann integral \( \int_a^b f(x) \, dx \).

- Partition the interval \([a, b]\) as \( P: a = x_0 < x_1 < x_2 < \cdots < x_n = b \).
- Define \( m_i \) to be the infimum, and \( M_i \) the supremum, of \( f \) on the \( i \)th subinterval in the partition.
- Form the sums \( U(f, P) = \sum_{i=1}^n M_i(x_i - x_{i-1}) \) and \( L(f, P) = \sum_{i=1}^n m_i(x_i - x_{i-1}) \).
- If \( \inf_P U(f, P) = \sup_Q L(f, Q) \) where the inf and the sup are taken over all partitions \( P \) and \( Q \) of \([a, b]\) then this common value is the value of the Riemann integral.

There are some pretty obvious problems with the Riemann integral.

- There are fairly easily constructed functions that are not Riemann integrable.
- It is possible to have \( f_n \) integrable for each \( n \), \( f_n \to f \) pointwise, but \( f \) not Riemann integrable.
- In the definition of the Riemann integral we are essentially approximating \( f(x) \) from above and below by step functions, and hoping that the integrals of these step functions can be used to approximate the integral of \( f(x) \).

The Lebesgue approach.

- Rather than partition the domain of integration \([a, b]\), partition the range of the function \( f([a, b]) \) using some partition \( P \).
- Then the collection of sets \( f^{-1}(P) \) partitions the domain \([a, b]\) into disjoint sets, not necessarily intervals.
- We approximate \( f(x) \) by functions that are constant on the “induced partition” of \([a, b]\). Such functions are called simple functions.
- These approximations are more versatile because the partition takes into account the structure of the function and not just the structure of the domain of integration.
- At a minimum, the sets in this induced partition of \([a, b]\) must be measurable, and leads us to our definition of measurable function.
B. Measurable Functions.

Definition 0.1 The real-valued function \( f \) defined on \( E \subseteq \mathbb{R} \) is \textit{measurable} if for all \( c \in \mathbb{R} \),
\[
\{ x \in E : f(x) > c \} = f^{-1}((c, \infty))
\]
is measurable.

Proposition 0.1 In the above definition, we can replace \( \{ x \in E : f(x) > c \} \) by \( \{ x \in E : f(x) \geq c \} \), \( \{ x \in E : f(x) < c \} \), or \( \{ x \in E : f(x) \leq c \} \).

Proposition 0.2 \( f \) is measurable if and only if for every open set \( \mathcal{O} \subseteq \mathbb{R} \), \( f^{-1}(\mathcal{O}) \) is measurable. From this it follows that if \( f \) is continuous on its measurable domain, then \( f \) is measurable.

Proposition 0.3 (i) The modification of a measurable function on a set of measure zero is measurable.
(ii) The restriction of a measurable function to a measurable subset of its domain is measurable.
(iii) If \( f \) and \( g \) are measurable with disjoint domains, then \( f \cup g \) is measurable.
(iv) Linear combinations of finite collections of measurable functions, each of which is finite almost everywhere, are measurable.
(v) Products of finite collections of measurable functions, each of which is finite almost everywhere, are measurable.
(vi) If \( g \) is measurable and \( f \) is continuous, then \( f \circ g \) is measurable.
(vii) If \( f \) is measurable, then so is its positive part \( f^+ \) and its negative part \( f^- \). Consequently \( |f| \) is also measurable.

Examples:
(1) If \( f \) and \( g \) are measurable, then \( f \circ g \) need not be measurable.
(2) If \( f \) is measurable and \( g \) is continuous, then \( f \circ g \) need not be measurable.