Nonmeasurable sets and the Cantor set.

A. Construction of a Nonmeasurable set.

Definition 0.1 Define the relation $\sim$ on $\mathbf{R}$ by $x \sim y$ if and only if $x - y$ is rational.

Proposition 0.1 The relation $\sim$ defined above is an equivalence relation on $\mathbf{R}$.

Remark 0.1 (1) Because $\sim$ is an equivalence relation, it partitions $\mathbf{R}$ into disjoint subsets. Moreover, if E is any subset of $\mathbf{R}$, then $\sim$ defines an equivalence relation on the set E, and hence defines a partition of E.

(2) Given a set E with $m^*(E) > 0$, define $\mathcal{C}_E$ to be a choice of representatives of the equivalence classes of E induced by $\sim$. Then clearly $\mathcal{C}_E \subseteq E$ and we will show that $\mathcal{C}_E$ is not measurable. We can assume without loss of generality that E is bounded.

Lemma 0.1 If $\Lambda \subseteq \mathbf{Q}$ then the collection $\{\mathcal{C}_E + \lambda\}_{\lambda \in \Lambda}$ is pairwise disjoint.

Lemma 0.2 If $\mathcal{C}_E$ is measurable, then $m(\mathcal{C}_E) = 0$.

Lemma 0.3 $m^*(E) \leq m^*(\mathcal{C}_E)$. Hence $\mathcal{C}_E$ is not measurable.

Remark 0.2 (1) Since E was an arbitrary set of positive outer measure, we have recovered the result of Vitali that says that any set of positive outer measure contains a nonmeasurable subset.

(2) The existence of a nonmeasurable set implies that there exist disjoint sets A and B such that $m^*(A \cup B) < m^*(A) + m^*(B)$.

B. The Cantor Set.

Definition 0.2 Let $C_0 = [0, 1]$. Define $C_1 = [0, 1/3] \cup [2/3, 1]$, $C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$, and in general define C_k to be the set obtained by removing the open middle third of each of the intervals making up the set C_{k-1}. Define the set $\mathbf{C}$ (the *Cantor middle-thirds set*) by $\mathbf{C} = \cap_{k=1}^{\infty} C_k$.

Remark 0.3 The following hold.

- Each C_k is the union of 2^k closed bounded intervals each of length $1/3^k$.
- $\mathbf{C}$ is closed and bounded. In fact it is a Borel set and hence measurable.
- $\mathbf{C}$ consists of all those points in $[0, 1]$ that can be written

  $$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n}$$

  where $a_n = 0$ or 2. Hence $\mathbf{C}$ is an uncountable set.
- $m(\mathbf{C}) = 0$.

Definition 0.3 For each k, define the function F_k on $[0, 1]$ as follows. Let $[0, 1] \sim C_k = \cup_{n=1}^{2^k-1} I_n$ where I_n is the n^{th} (counting from the left) open interval making up the complement of C_k in $[0, 1]$. Define $F_k(x) = n/2^k$ for $x \in I_n$, and linear otherwise in such a way that $F_k(0) = 0$, $F_k(1) = 1$ and F_k is continuous on $[0, 1]$. (This is easier to draw than to describe.)

Lemma 0.4 For each natural number k, $|F_{k+1}(x) - F_k(x)| \leq 2^{-k}$ for all $x \in [0, 1]$. Hence the sequence $\{F_k\}$ converges uniformly on $[0, 1]$ to a continuous function $\varphi(x)$ called the *Cantor-Lebesgue function.*

$\varphi(x)$ satisfies the following properties.

(1) $\varphi(0) = 0$ and $\varphi(1) = 1$.

(2) $\varphi(x)$ is nondecreasing on $[0, 1]$.

(3) At each point x of the open set $[0, 1] \sim \mathbf{C}$, $\varphi(x)$ is differentiable and $\varphi'(x) = 0$.