Alternate Characterization of Measurability.

A. Excision Property.

Proposition 0.1 Let $E \subseteq \mathbf{R}$. Given $\epsilon > 0$, there exists an open set $\mathcal{O}$ such that $E \subseteq \mathcal{O}$ and $m^*(\mathcal{O}) < m^*(E) + \epsilon$.

Proposition 0.2 A set $E \subseteq \mathbf{R}$ is measurable if and only if given $\epsilon > 0$, there exists an open set $\mathcal{O}$ such that $E \subseteq \mathcal{O}$ and $m^*(\mathcal{O} \sim E) < \epsilon$.

Remark 0.1 (1) Note that since the result of the first proposition always holds, then the measurability criterion in the previous proposition will hold as long as for sets $\mathcal{O}$ and E,

$$m^*(\mathcal{O} \sim E) \leq m^*(\mathcal{O}) - m^*(E).$$

(2) But by the subadditivity of outer measure we always have that

$$m^*(\mathcal{O}) \leq m^*(\mathcal{O} \sim E) + m^*(E),$$

and hence that

$$m^*(\mathcal{O} \sim E) \geq m^*(\mathcal{O}) - m^*(E).$$

(3) Therefore the measurability criterion in the previous proposition holds if we can show that for sets $\mathcal{O}$ and E,

$$m^*(\mathcal{O} \sim E) = m^*(\mathcal{O}) - m^*(E).$$

This is called the *excision property* of measurable sets.

Proof: ($\Longrightarrow$) If E is measurable then the excision property follows immediately from the Carathéodory property.

($\Longleftarrow$) We want to show that any set that can be approximated from without by an open set as in the proposition satisfies the Carathéodory property. We know from previous results that the collection of sets that satisfy the Carathéodory property form a σ-algebra containing the Borel sets, and that any set with zero outer measure satisfies the Carathéodory property. From these observations the result follows.

Proposition 0.3 A set $E \subseteq \mathbf{R}$ is measurable if and only if given $\epsilon > 0$, there exists a closed set F such that $F \subseteq E$ and $m^*(E \sim F) < \epsilon$.

Proposition 0.4 Let $E \subseteq \mathbf{R}$ be measurable with finite outer measure. Then given $\epsilon > 0$, there is a finite disjoint collection of open intervals $\{I_k\}_{k=1}^n$ such that

$$m^*\left(E \Delta \left(\bigcup_{k=1}^n I_k\right)\right) < \epsilon.$$

B. Lebesgue Measure.

Definition 0.1 *Lebesgue measure*, denoted by m, is the restriction of outer measure to the σ-algebra of Lebesgue measurable sets.

Proposition 0.5 Lebesgue measure is countably additive.

Proposition 0.6 Let $\{A_n\}$ be an ascending sequence of measurable sets, that is, $A_n \subseteq A_{n+1}$ for all n. Then

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m(A_n).$$

Let $\{B_n\}$ be an descending sequence of measurable sets, that is, $B_{n+1} \subseteq B_n$ for all n. Then

$$m\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n\to\infty} m(B_n).$$

Borel-Cantelli Lemma. Let $\{E_k\}$ be a countable collection of measurable sets such that $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost every $x \in \mathbf{R}$ belongs to at most finitely many of the E_k.