Lebesgue Outer Measure and Lebesgue Measure.

A. Basic notions of measure.

Our goal is to define a *set function* m defined on some collection of sets and taking values in the nonnegative extended real numbers that generalizes and formalizes the notion of length of an interval. Such a set function should satisfy certain reasonable properties

- m is defined on a sufficiently rich σ-algebra.
- $m(I) = \ell(I)$ where I is any interval, and $\ell(I)$ denotes the length of I.
- m is translation invariant.
- m is countably additive.

Remark 0.1 Countable additivity is important because it implies that m is (1) monotonic and (2) that $m(\emptyset) = 0$ as long as m is not identically infinity.

B. Outer Measure.

Definition 0.1 Let $E \subseteq \mathbf{R}$. The *(Legesgue) outer measure* of E, denoted $m^*(E)$ is defined to be

$$m^*(E) = \inf \left\{ \sum_{k=1}^{\infty} \ell(I_k) \right\}$$

where the infimum is taken over all countable collections of open intervals $\{I_k\}$ with the property that $E \subseteq \bigcup_{k=1}^{\infty} I_k$.

Remark 0.2 (1) Outer measure is defined for every subset of $\mathbf{R}$.
(2) Outer measure is monotonic, that is, if $A \subseteq B$ the $m^*(A) \leq m^*(B)$. This is because any covering of B by open intervals is also a covering of A so that the latter infimum is taken over a larger collection than the former.
(3) $m^*(\emptyset) = 0$, and m^* is translation invariant.

Example. If C is a countable set then $m^*(C) = 0$. In particular, the outer measure of the rational numbers is zero.

Proposition 0.1 The outer measure of any interval is its length.

Proof: It is sufficient to prove this result for closed bounded intervals of the form $[a, b]$, a, $b \in \mathbf{R}$, for if we know the result for such intervals then (1) we know it for unbounded intervals, for if I were such an interval, then given any number $M > 0$ there is a closed bounded interval $J \subseteq I$ such that $\ell(J) \geq M$ and by the monotonicity of outer measure, $m^*(I) \geq m*(J) = \ell(J) \geq M$, so that $m^*(I) = \infty$, and (2) we know it for arbitrary bounded intervals.

Proposition 0.2 Outer measure is countably subadditive, that is, if $\{E_k\}$ is any countable collection of subsets of $\mathbf{R}$, then

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \le \sum_{k=1}^{\infty} m^*(E_k).$$

C. Measurable Sets.

Definition 0.2 A set $E \subseteq \mathbf{R}$ is said to be *(Legesgue) measurable* provided that for any set A,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^C).$$

Remark 0.3 (1) The criterion in the previous definition is sometimes called the *Carathéodory criterion.*

(2) Outer measure is countably subadditive but is not countably additive, and indeed there are disjoint sets A and B such that $m^*(A \cup B) < m^*(A) + m^*(B)$. What the Carathéodory criterion says is that a set is measurable if and only if it can be used to split any set A into two disjoint pieces for which outer measure is additive.

(3) Since m^* is countably subadditive, and since for any sets A and E, $A = (A \cap E) \cup (A \cap E^C)$, always $m^*(A) \le m^*(A \cap E) + m^*(A \cap E^C)$. Hence to show that a set E is measurable we have only to show that $m^*(A) \ge m^*(A \cap E) + m^*(A \cap E^C)$ for every A.

(4) It is clear that if E is measurable then so is E^C.

(5)

Proposition 0.3 If $m^*(E) = 0$ then E is measurable.

Proposition 0.4 Let $\mathcal{M}$ denote the collection of all Lebesgue measurable subsets of $\mathbf{R}$. Then $\mathcal{M}$ is a σ-algebra that contains the σ-algebra of Borel sets.