Sigma Algebras and Borel Sets.

A. $\sigma-$Algebras.

Definition 0.1 A collection $\mathcal{A}$ of subsets of a set $\mathcal{X}$ is a *σ-algebra* provided that (1) $\emptyset \in \mathcal{A}$, (2) if $A \in \mathcal{A}$ then its complement is in $\mathcal{A}$, and (3) a countable union of sets in $\mathcal{A}$ is also in $\mathcal{A}$.

Remark 0.1 It follows from the definition that a countable intersection of sets in $\mathcal{A}$ is also in $\mathcal{A}$.

Definition 0.2 Let $\{A_n\}_{n=1}^{\infty}$ belong to a sigma algebra $\mathcal{A}$. We define

$$\limsup\{A_n\} = \bigcap_{k=1}^{\infty}\left[\bigcup_{n=k}^{\infty} A_n\right],$$

and

$$\liminf\{A_n\} = \bigcup_{k=1}^{\infty}\left[\bigcap_{n=k}^{\infty} A_n\right].$$

Remark 0.2 (1) $\limsup\{A_n\}$ is the set of points that are in infinitely many of the A_n, and $\liminf\{A_n\}$ is the set of points that fail to be in at most finitely many of the A_n, in other words $x \in \liminf\{A_n\}$ if and only if there is an index k such that $x \in A_n$ for all $n \geq k$.

(2) Recall that if $\{x_n\}$ is a bounded sequence of real numbers, then

$$\limsup_{n\to\infty}\{x_n\} = \lim_{n\to\infty}\sup_{n\geq k} x_n = \inf_n \sup_{n\geq k} x_n$$

because the sequence $y_k = \sup_{n\geq k} x_n$ is nonincreasing and bounded below. Also

$$\liminf_{n\to\infty}\{x_n\} = \lim_{n\to\infty}\inf_{n\geq k} x_n = \sup_n \inf_{n\geq k} x_n$$

because the sequence $y_k = \inf_{n\geq k} x_n$ is nondecreasing and bounded above.

If we partially order the sets in the σ-algebra $\mathcal{A}$ by inclusion, then for any sequence $\{A_n\}$ of sets,

$$\sup\{A_n\} = \bigcup_{n=1}^{\infty} A_n, \quad \textit{and} \quad \inf\{A_n\} = \bigcap_{n=1}^{\infty} A_n.$$

With this notation,

$$\limsup\{A_n\} = \inf\left[\sup\{A_n\}_{n=k}^{\infty}\right]$$

and

$$\liminf\{A_n\} = \sup\left[\inf\{A_n\}_{n=k}^{\infty}\right]$$

in analogy with the definition for sequences of real numbers.

(3) In further analogy with the situation for sequences of real numbers, we have the following propositions.

Proposition 0.1 Let $\{x_n\}$ be a sequence of real numbers and let $A_n = (-\infty, x_n)$. Then

$$\limsup\{A_n\} = (-\infty, x) \quad where \quad x = \limsup\{x_n\}$$

and

$$\liminf\{A_n\} = (-\infty, x) \quad where \quad x = \liminf\{x_n\}.$$

Proposition 0.2 $\liminf\{A_n\} \subseteq \liminf\{A_n\}$.

B. Borel Sets.

Definition 0.3 A set $E \subseteq \mathbf{R}$ is an F_σ set provided that it is the countable union of closed sets and is a G_δ set if it is the countable intersection of open sets. The collection of *Borel sets*, denoted $\mathcal{B}$, is the smallest σ-algebra containing the open sets.

Remark 0.3 (1) Every G_δ set is a Borel set. Since the complement of a G_δ set is an F_σ set, every F_σ set is a Borel set.
(2) Every interval of the form $[a, b)$ is both a G_δ set and an F_σ set and hence is a Borel set. In fact, the Borel sets can be characterized as the smallest σ-algebra containing intervals of the form $[a, b)$ for real numbers a and b.

C. Example: Problem 44, Section 1.5.

Claim: Let p be a natural number, $p > 1$, and $x \in [0, 1]$. Then there is a sequence of integers $\{a_n\}$ where $0 \leq a_n < p$ and such that

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}.$$

This expansion is unique except when $x = q/p^n$ for some natural number q in which case there are exactly two such expansions.

Proof: What follows in an outline of the proof of the Claim. For $0 \leq k < p$ define $I_{1,k} = [k/p, (k+1)/p]$. Clearly the intervals $I_{1,k}$ are essentially disjoint in the sense that they overlap in at most one point, and their union is $[0, 1]$. Define a_1 to be a number such that $x \in I_{1,a_1}$. The choice of a_1 is unique except when $x = k/p$ for some $1 \leq k < p$. If we choose $a_1 = k$ and the sequence required by the Claim is $\{k, 0, 0, \ldots\}$. If we choose $a_1 = k - 1$ then the sequence required by the Claim is $\{k-1, p-1, p-1, \ldots\}$. (Note that for any natural number k, $\sum_{n=k}^{\infty} \frac{p-1}{p^n} = \frac{1}{p^{k-1}}$. This explains why the second sequence works.)

In the case where there is no ambiguity in the choice of a_1, define $y = x - a_1/p$ and the intervals $I_{2,k} = [k/p^2, (k+1)/p^2]$ for $0 \leq k < p$. Choose a_2 so that $y \in I_{2,a_2}$. Ambiguity will occur only if $y = k/p^2$ for some $1 \leq k < p$ and in this case the sequences required are either $\{a_1, k, 0, 0, \ldots\}$ or $\{a_1, k-1, p-1, p-1, \ldots\}$.

In general we proceed as follows. Assuming that there was no ambiguity in the choice of a_1, a_2, $\ldots$, a_{n-1}, define $y = x - \sum_{k=1}^{n-1} a_k/p^k$, and the intervals $I_{n,k} = [k/p^n, (k+1)/p^n]$ for $0 \leq k < p$. Choose a_n so that $y \in I_{n,a_n}$. Ambiguity occurs only when $y = k/p^n$ for some $1 \leq k < p$, and in this case the sequences required are either $\{a_1, \ldots, a_{n-1}, k, 0, 0, \ldots\}$ or $\{a_1, \ldots, a_{n-1}, k-1, p-1, p-1, \ldots\}$.

It remains to show that in fact $x = \sum_{k=1}^{\infty} a_k/p^k$, and that the converse of the claim is true.

Remark 0.4 (1) If $p = 10$ then the expansion given above is the familiar decimal expansion of a number. If $p = 2$ the expansion is called the binary expansion, and if $p = 3$ the ternary expansion.

(2) Later on we will construct the *Cantor set* C, a set with unusual and interesting properties. By comparing the construction of C with the above problem, it can be seen that C consists of all points in $[0, 1]$ that have a ternary expansion such that $a_n \neq 1$ for all n.